JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and ornament thereunto."—BACON.

VOL. LIII.

LONDON:
CHARLES AND EDWIN LAYTON,
FARRINGDON STREET, E.C. 4.

PARIS: 8, RUE LAMARTINE, 8. MELBOURNE: McCARRON, BIRD & CO.
NEW YORK: THE SPECTATOR COMPANY.

1922.

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LONDON:
PRINTED BY CHARLES AND EDWIN LAYTON,
FARRINGDON STREET, E.C. 4.

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An Investigation into the mortality experienced by life tenants under reversions, with some conclusions drawn therefrom. By E. H. LEVER, F.I.A., *of the Prudential Assurance Company.*

[Submitted to the Institute, 31 October 1921.]

(1). THE necessity for a reliable investigation into the mortality experienced by life tenants under reversions has long been felt.

So far as I am aware, the only published experience is that given by Mr. Neil Campbell in a paper read before the Faculty of Actuaries in 1902 (T.F.A., vol. i, p. 79), although doubtless individual offices have at various times examined the mortality in respect of their own purchases. The investigation by Mr. Campbell was of considerable service in indicating the trend of the mortality, but was unfortunately of limited value inasmuch as it was based upon only 226 lives producing 2,097 years of exposure to risk.

The present investigation is of a much more extensive character and will I hope be found valuable.

Seventeen Life Assurance Companies and four Reversionary Companies were good enough to furnish particulars of their cases and a substantial body of data was in this way obtained.

I wish to take this opportunity of placing on record my great indebtedness to these companies for the generous way in which each one approached willingly placed information at my disposal even when not a little inconvenience and labour was entailed.

Except for certain companies in the north of England and in Scotland, the companies approached comprise all those that purchase reversions on any substantial scale.

Character of Data and Preliminary Treatment.

(2). The investigation was limited to life tenants in respect of absolute and contingent reversions definitely purchased by the offices concerned.

This limitation was decided upon after due deliberation, and contributory causes to the decisions were:

(*a*) that by so doing it was more likely that the data would be homogeneous.

(*b*) that it simplified the work for the companies.

(*c*) that to include loans as well as purchases was undesirable, since, in the case of loans, certain vital information, such as the date of death, was much less likely to be reliable.

(*d*) that in the case of life interests, except where these are contingent, mortality is comparatively unimportant from the point of view of the purchaser, the main consideration being whether the life is assurable and, if so, at what rate.

Duplicates.

(3). Duplicates were entirely eliminated and the experience based on lives only, except when the exposure due to one purchase terminated before the next purchase was made. Only one instance of this occurred.

In the case of overlapping periods of risk, cards relating to the same life were combined and the date of purchase taken as the earliest date. In practically every instance this resulted simply in discarding all cards except that relating to the first purchase, since from the nature of the case it almost invariably happened that the date and cause of exit were the same in respect of all purchases in which the same life was involved.

As might have been expected, the number of duplicates eliminated was considerable, amounting to about 30 per-cent of the cards sent in. In one instance there were as many as 16 purchases in respect of the same reversion.

Contingent Reversions—Definition.

(4). Reversions were deemed to be contingent when the receipt of the fund by the reversioner depended upon his surviving the life tenant or tenants. Cases otherwise absolute but involving some minor contingency, such as an issue risk, were treated as absolute for the purposes of the investigation.

Date of Purchase.

(5). In most cases the date of purchase was not open to question, but two types called for special consideration :

(*a*) purchases involving an option to redeem,

(*b*) purchases by foreclosure of mortgage.

In type (*a*) the date of purchase was taken as the date the option expired ; all cases falling in or existing during the option period were excluded whether the option was exercised or not.

Only a few instances occurred of type (*b*) and in these the date of entry was taken as the date of foreclosure and not the date of original loan.

Reliability of Data.

(6). For the most part the companies were able to furnish exact dates of birth, purchase and exit.

Even where the information supplied was incomplete, enough was generally available to enable the card to be accurately dealt with, and in the remaining cases common sense assumptions were made for the purpose of arriving as nearly as possible at a correct result.

Period Covered by Investigation and Extent of Data.

(7). The investigation was limited to the period 1905 to 1919, cases being traced from their anniversary of purchase in 1905, if they were purchased before that year, and from the date of purchase, if acquired subsequently, to their purchase anniversary in 1919 or previous exit.

The total number of cards received was 5,042
Of which there remained after discarding duplicates and making other necessary eliminations 3,463

comprising 2,805 females
and 658 males.

The following is a summary of the data, absolute and contingent reversions combined :

Females.

Number of cases included	Existing	Other Exits	Deaths	Total years of Exposure to Risk
2,805	1,475	48	1,282	23,226

Males.

Number of cases included	Existing	Other Exits	Deaths	Total years of Exposure to Risk
658	276	19	363	4,622

It will be seen that the number of cases passing out of observation other than by death or existing were very few.

Of the 2,805 reversions involving female life tenants 2,473 were absolute and 332 contingent, while of the 658 reversions involving male life tenants 540 were absolute and 118 contingent.

TABULATION OF DATA AND METHOD OF OBTAINING EXPOSED TO RISK.

(8). The character of the data was such that it was possible to obtain the numbers exposed to risk with a minimum degree of error.

The method adopted was simple in character and involved nothing new in principle. It can best be shown symbolically.

Let $[x]$ represent assumed exact age at purchase. This was taken as the nearest integral age at the date purchase was made.

Let t represent the number of years elapsed since purchase.

Let $n_{[x]}$ represent the number of entrants coming under observation at date of purchase.

Let $s_{[x]+t}$ represent the number of entrants coming under observation in 1905 on the tth purchase anniversary.

Let $d_{[x]+t}$ represent the number of deaths during year of duration $(t-1)$ to (t) and consequently passing out of observation on the tth purchase anniversary.

Let $e_{[x]+t}$ represent the number of cases existing on their tth purchase anniversary in 1919.

Let $w_{[x]+t}$ represent the number of cases, other than deaths or existing, passing from observation on the tth purchase anniversary. The assumed exact duration at exit was taken in these cases as the nearest integral duration at date of exit.

A complete tabulation of the data in select form was made, in accordance with the above principles. "Old" and "New" reversions were tabulated separately and within each of these two groups a distinction was made between absolute and contingent reversions.

The number exposed to risk at each age and year of duration was then found by the application of the following formulas:

"Old" Reversions.

$$E_{[x]+t} = E_{[x]+t-1} + \{s - d - w - e\}_{[x]+t}$$

"New" Reversions.

$$E_{[x]} = n_{[x]} - w_{[x]+0}$$

$$E_{[x]+t} = E_{[x]+t-1} - (e + d + w)_{[x]+t}$$

where $E_{[x]+t}$ represents the number exposed to risk in the year of duration t to $t+1$ in respect of reversions purchased at age $[x]$.

It follows from the above that $q_{[x]+t} = \dfrac{d_{[x]+t+1}}{E_{[x]+t}}$.

Preliminary Tables.

(9). As it was desired completely to investigate the effect, if any, of selection, three aggregate tables were formed from the female data, namely, (*a*) full aggregate, (*b*) truncated aggregate excluding the first five years following purchase, and (*c*) truncated aggregate excluding the first ten years following purchase. The unadjusted data for these three tables are given in Table A, together with the unadjusted full aggregate data for males.

Before proceeding to analyse these results there are one or two minor points which are worth placing on record.

(*a*) The average age at purchase of all "new" reversions, absolute and contingent combined, was 64·8 years in the case of females and 64·6 years in the case of males and over 85 per-cent of these reversions were purchased between ages 50 and 80.

(*b*) The average age at purchase of contingent reversions was about three years higher than that of absolute reversions and this feature persisted in all sections of the data.

(*c*) The distribution between "old" and "new" data was as follows:

Females.—Full Aggregate.

Class of Data	No. of Cases	Total years of Exposure to Risk
Old New	1,391 1,414	13,010 10,216
Total	2,805	23,226

Males.—Full Aggregate.

Class of Data	No. of Cases	Total years of Exposure to Risk
Old New	275 383	2,262 2,360
Total	658	4,622

The Effect of Selection.

(10). The following tables show in respect of the three female aggregate tables :

(*a*) the unadjusted rates of mortality,

(*b*) the unadjusted expectations of life,

in each case the mean of three values being taken.

(*a*) *Females.—Unadjusted Values of* q_x.

Central Age of Group	Full Aggregate	Truncated Aggregate	
		Excluding first 5 years	Excluding first 10 years
51	·0076	·0095	·0000
54	·0181	·0151	·0196
57	·0157	·0106	·0145
60	·0111	·0097	·0098
63	·0183	·0223	·0191
66	·0210	·0182	·0147
69	·0320	·0321	·0375
72	·0523	·0550	·0594
75	·0566	·0541	·0569
78	·0779	·0777	·0873

(*b*) *Females.—Unadjusted Values of e_x.*

Central Age of Group	Full Aggregate	Truncated Aggregate	
		Excluding first 5 years	Excluding first 10 years
57	20·00	20·28	20·08
60	17·80	17·77	17·61
63	15·50	15·47	15·40
66	13·35	13·35	13·03
69	11·38	11·31	10·92
72	9·62	9·63	9·31
75	8·17	8·18	8·04
78	6·61	6·59	6·53
81	5·45	5·39	5·53
84	4·37	4·32	4·48

Neither of these tables gives any marked indication of the existence of selection on the part of vendors of reversions. Considering that the data were reasonably extensive, the differences between the "full aggregate" and the "excluding first five years" values are remarkably small and are moreover not always in the same direction, the rates of mortality in the latter table being, for a considerable section of ages, actually lighter than in the former.

One curious feature worth noticing is that, speaking generally, the mortality during the second five years following purchase was lighter than during the first five years. Whether this result was due to some direct cause, such as the presence of a proportion of abnormally bad lives, and would occur again in a subsequent investigation, or whether it was simply due to a certain paucity of data at the later durations, is a matter for discussion, but its existence is beyond doubt. In this connection the following summary of the data will give an idea of the relative weight to be attached to each of the preliminary tables.

Females.

Table	Total years of Exposure to Risk	Total Deaths
Full aggregate	23,226	1,282
Excluding first 5 years	16,043	1,000
Excluding first 10 years	9,666	720

In any case Tables (*a*) and (*b*) revealed such an absence of definite evidence of selection that it was clear, even at this stage, that one could proceed profitably to the completion and graduation of the full aggregate table, and this was accordingly done: the method of graduation is described subsequently.

(11). With the graduated table available it was possible still further to investigate the influence, if any, of selection and the following additional tests were therefore imposed.

The numbers exposed to risk and the concomitant deaths in each of the first 10 years of duration were separately ascertained and arranged according to age. The expected deaths by the graduated full aggregate table were then calculated for each of these ten separate sections of data and the total of such expected deaths was compared with the total number of deaths that actually occurred. The result was very instructive and is given in the following table :

Females.

Year of Duration	Deaths		$100\frac{(2)}{(3)}$	Quinquennium of Duration	Deaths		$100\frac{(6)}{(7)}$
	Actual	Expected			Actual	Expected	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0–1	44	53·38	82·4	1	282	284·95	99·0
1–2	63	56·37	111·8				
2–3	54	56·01	96·4				
3–4	54	58·87	91·7				
4–5	67	60·32	111·1				
5–6	54	58·76	91·9	2	280	295·31	94·8
6–7	59	59·42	99·3				
7–8	60	60·03	100·0				
8–9	63	58·31	108·0				
9–10	44	58·79	74·8				
Total	562	580·26	96·9	...	562	580·26	96·9

An examination of the ratios in column (4) shows that except for the first and tenth years the expected and actual deaths agree very closely indeed and that, moreover, far from there being any evidence that the actual deaths are consistently below those expected, the ratio fluctuates fairly regularly above and below 100 per-cent.

The ratios in column (8) exhibit clearly the lighter mortality in the second quinquennium which has already been remarked. It is interesting to notice that this is entirely due to an abnormally light mortality in the tenth year, and this fact is, I think, in itself sufficient good ground for assuming that it is a feature which is not likely to recur.

It will be seen that taking the first 10 years as a whole the actual deaths were 96·9 per-cent of those expected by the full aggregate table. If the tenth year, in which the mortality was abnormally low, be omitted the actual deaths amounted to 518 against 521·47 expected or 99·3 per-cent.

The magnitude of these ratios depends of course in a measure upon the extent to which the data relating to the first 10 years enter into the composition of the full aggregate rates, but even allowing for this, the only conclusion to be drawn from the above table, especially in view of the absence of any indication of progressive general deterioration of vitality with an increase of duration, is that, taking the data as a whole, selection had no practical effect.

In some respects it would have been better to have based the foregoing analysis on the truncated aggregate (excluding first 10 years) table but, unfortunately, the data of this table at the earlier ages were hardly sufficient for reliable results. For an analysis of the first five years only, the truncated aggregate (excluding the first 5 years) could have been employed, but the rates by this table are so nearly the same as those of the full aggregate table that the results would have been almost identical, and in any event for an analysis of the second five years both tables labour under the same disadvantage.

For the benefit of those to whom it makes an appeal, however, I give in Appendix B a comparison between the actual deaths experienced in each of the first 10 years and those expected by the truncated aggregate (excluding first 10 years) table.

(12). It is not without interest to investigate separately the mortality among cases involving an option to repay; it will be remembered that in these cases the date of purchase was taken as the date upon which the right to exercise the option expired.

A priori, it would not appear that selection, if any, would be as strong as in the case of an ordinary purchase, but the point is worth investigation.

Unfortunately the numbers were too small for really reliable conclusions to be drawn, there being only 113 option cases among the female lives, but the following facts resulting from an analysis of the mortality among these cases are worth examination.

Females.—Purchases with option to repay.

Duration (1)	Expected Deaths by Female full aggregate table (2)	Actual Deaths (3)	$100\frac{(3)}{(2)}$ (4)
First 5 years ...	15·90	24	151
Second 5 years ...	9·95	13	131
Over 10 years ...	14·84	15	101
...	40·69	52	128

The excessive mortality shown in the early years of duration is, in spite of the inadequacy of the data, somewhat remarkable, especially as in point of fact the whole of the excess in the first quinquennium occurred in the first and second years of duration. One would at least have expected to find the mortality in the first year a little lighter than normal, inasmuch as if the life tenant were actually ill at the expiry of the option period, the borrower would naturally use every effort to find the money to repay. While therefore it does appear that the mortality of life tenants in option cases is somewhat heavier than that of the main body, one can only assume that the distribution of the excess over years of duration in the few cases that have been examined is largely accidental, and due to paucity of data, and is not an indication of a definite law. In any event there was certainly no selection against lenders.

(13). In order to make as complete as possible the justification for using a full aggregate table only and for the construction of monetary tables based thereon yet another comparison was made. This time the numbers exposed to risk were calculated separately for the "new" and the "old" data, the expected deaths by the graduated full aggregate table obtained for each of these two divisions, and the totals compared with the corresponding totals of the actual deaths. As 92 per-cent of the years of exposure to risk in the case of the "new" data were within the first 10 years of assurance, and only 32 per-cent in the case of the "old" data, it was reasonable to suppose that if selection had taken place in any marked degree it would be reflected in correspondingly marked differences between the actual and expected deaths in each of the two sections of the data under consideration.

The following, however, was the actual position:

Females.

Class of Data	Deaths		$100\frac{(2)}{(3)}$
	Actual	Expected	
(1)	(2)	(3)	(4)
New	421	435·6	96·6
Old	861	847·2	101·6

These ratios bear out the more detailed results in paragraph (11) and taken in conjunction with all the other facts, add further

weight to the conclusion that the construction of select tables would be unnecessary labour.

The case against the use of select tables being thus complete, attention may now be directed more particularly to the actual rates of mortality according to the full aggregate tables, male and female.

Graduation of Full Aggregate Tables.

Females. (14). As will be apparent from Table A and Diagram 1, the ungraduated rates for this table ran very irregularly.

After a certain amount of experiment, the graduation finally adopted was that of the application of Makeham's formula. The value of $\log_{10}c$ was determined approximately from the facts themselves and was taken as ·045.

The values of the remaining constants are given at the foot of Table B and the following comparisons between the expected and actual deaths, and between the graduated and ungraduated expectations of life, will enable judgment to be passed as to whether the graduated results can be taken as representing as nearly as possible the true law underlying the observed facts.

Females.—Full Aggregate.

Comparison between Actual and Expected Deaths.

Age Group (1)	Exposed to Risk (2)	Actual Deaths (3)	Expected Deaths (4)	Expected Deviation (5)	Actual Deviation (6)		Accumulated Deviation (7)	
					+	−	+	−
30–34	15	...	·08	·25				
35–39	50	...	·29	·44	...	·38	...	·38
40–44	189	2	1·25	·91				
45–49	475	8	3·72	1·55	...	4·28	...	4·66
50–54	1,020	9	10·18	2·55	1·18	...	...	3·48
55–59	2,032	34	27·34	4·16	...	6·66	...	10·14
60–64	3,300	54	63·47	6·33	9·47	...	...	·67
65–69	4,203	113	120·81	8·67	7·81	...	7·14	...
70–74	4,150	195	184·26	10·60	...	10·74	...	3·60
75–79	3,568	247	249·97	12·18	2·97	...	...	·63
80–84	2,516	286	278·29	12·57	...	7·71	...	8·34
85–89	1,284	198	223·57	10·91	25·57	...	17·23	...
90–94	365	111	95·98	6·70	...	15·02	2·21	...
95–99	56	23	21·87	2·92	...	1·13	1·08	...
100–up	3	2	1·68	·69	...	·32	·76	...
Total	23,226	1,282	1,282·76	±81·43	47·00	46·24	...	...
					$\underbrace{\qquad}_{\pm 93 \cdot 24}$			

Females.—Full Aggregate.

Comparison between Graduated and Ungraduated Expectations of Life.

Age Group	Mean of Five Values		Differences	
	Ungraduated	Graduated	−	+
50–54	23·27	23·44	...	·17
55–59	19·91	19·65	·26	...
60–64	16·25	16·05	·20	...
65–69	12·68	12·72	...	·04
70–74	9·66	9·73	...	·06
75–79	7·13	7·14	...	·01
80–84	5·08	5·01	·07	...
85–89	3·29	3·33	...	·04
90–94	1·98	2·08	...	·10

I do not propose to enter into a detailed criticism of these results since the graduation was, after all, only a means to an end and not the end itself.

Suffice to say that after an exhaustive examination I was satisfied that, although the graduation does not attain perfection in every respect, its adoption was fully justified, particularly as it brought with it the advantages incidental to Makeham's law. I must confess to a certain amount of surprise at finding a Makeham's curve so nearly to represent the female rates of mortality.

Males.

(15). A preliminary graduation of the male rates by means of a summation formula of considerable smoothing power and a graphical comparison of the resulting series with the graduated female rates pointed to the high probability that the male rates could be suitably represented by a Makeham curve with the same value of c as in the case of the females. In view of the advantages attaching to such a relation between male and female curves, a second graduation was made on this basis. The following comparison shows that the result was satisfactory and the Makeham graduation was therefore adopted.

Males.—Full Aggregate.

Age Group	Exposed to Risk	Actual Deaths	Expected Deaths	Expected Deviation	Actual Deviation +	Actual Deviation −	Accumulated Deviation +	Accumulated Deviation −
30–34	9							
35–39	22	0	2·14	2·20	2·14	...	2·14	...
40–44	24							
45–49	54							
50–54	194	9	4·52	1·68	...	4·48	...	2·34
55–59	395	10	11·10	2·63	1·10	...	...	1·24
60–64	678	27	24·08	3·53	...	2·92	...	4·16
65–69	829	40	39·67	4·92	...	·33	...	4·49
70–74	864	51	59·29	5·95	8·29	...	3·80	...
75–79	796	85	80·51	6·80	...	4·49	...	·69
80–84	488	69	74·19	6·34	5·19	...	4·50	...
85–89	217	54	49·57	4·95	...	4·43	·07	...
90–94	49	16	16·67	2·65	·67	...	·74	...
95 up	3	2	1·38	·69	...	·62	·12	...
...	4,622	363	363·12	±42·34	17·39	17·27	...	...
					±34·66			

The graduated and ungraduated rates of mortality are compared graphically in Diagram 2.

Comparison of Rates of Mortality with those of other Tables.—Females.

(16). The valuation of reversions in practice has, up to the present, been carried out mainly on the basis of the Carlisle and the $O^{[af]}$ Tables, and it is natural therefore that the first comparison should be with the rates of mortality according to each of these two experiences.

Owing to the relative absence of selection in the life tenants experience, comparison with the $O^{[af]}$ is a little difficult inasmuch as in the latter table selection was present in a marked degree. The two following hypotheses, which seem sufficiently sound in their logical structure, may however be employed to bridge the difficulty and to afford a basis of true comparison : (*a*) that where selection does not exist, the mortality experienced by a body of lives may be taken as representative of the whole class from which that body is drawn and (*b*) that where selection does exist the mortality of the whole class is most nearly represented by the " ultimate " mortality of the lives under observation, this being a measure of the mortality of the lives among whom the force of selection is no longer felt.

To determine therefore whether life tenants are, generally speaking, lives of the same class as annuitants, a comparison should be made between the full aggregate rates in the former case and the ultimate rates in the latter.

This is accordingly done in Diagram 3, which exhibits graphically the values of q_x by the Life Tenants (Female) Table, the Carlisle Table and the $O^{[af]}$ (ult.) Table. In the case of the Carlisle Table the ordinates have been moved up three years throughout in order to give effect to the general practice of rating down the age when this table is employed for female lives.

The Carlisle Table. It will be seen from Diagram 3 that the rates of mortality by the Carlisle Table do not represent either in magnitude or general progression the experience of life tenants; this is perhaps not surprising in view of the nature of the basic data and the method employed in constructing the former table. The continued use of the Carlisle Table cannot therefore be justified on any ground except that, thanks to the painstaking energy of the earlier actuaries, we are in possession of a collection of monetary functions based upon it which far exceed in number and variety any similar calculations in respect of more modern tables. The fact that the use of the table in the past has not led to considerable error is largely due to the fortuitous circumstance that there is an anomalous progression in the rates of mortality from age 91 onwards which counteracts the markedly higher mortality in the main body of the table.

Diagram 3 and the values of e_x given on page 23, moreover demonstrate quite clearly that the custom of making a uniform deduction of three years from the age in no way meets the situation; to produce anything like accurate results the deduction should vary considerably, being about 8 years at age 50 and 3 years at age 80. The possession of more definite knowledge on this point is, however, hardly an argument in favour of retaining the Carlisle Table as the instrument of measurement for the valuation of reversions, since precisely the same reasoning could be applied in favour of the employment of any table or even an arbitrarily constructed series of rates if it so happened that certain functions required had already been calculated on the basis of that table or series.

Altogether the time seems to have come for discarding the Carlisle Table in practical valuations and it is hoped that the present investigation will be of some service in this respect by

assisting to open up the route from rough approximation to scientific precision.

The $O^{[af]}$ Table.

The female rates of mortality are for the most part markedly lighter than those of the $O^{[af]}$ (ult.), the curves crossing between ages 94 and 95. The two curves have very much the same general character except that the one representing life tenants mortality runs up more steeply at the older ages, a fact that is otherwise evident from the relatively high value of log c used in the graduation.

So far as it goes therefore Diagram 3 shows that female life tenants are a better class of life even than office annuitants, although the evidence points not so much to the limit of life being higher as to more people living to the older ages.

Unfortunately the comparison takes no account of one important factor, namely the respective dates of the investigations, a factor which certainly cannot be ignored. Nearly 30 years have elapsed since the close of the period covered by the British Offices investigation, and it seems highly probable in view of the general improvement in vitality which other investigations show to be coincident with the passing of time, that a similar table constructed on recent data would present rates of mortality approaching much more closely those experienced by life tenants. The present investigation supplies, in fact, a very weighty argument in favour of the early construction of a new annuity table.

The Government Annuity (1910) Table.

In Diagram 4 two further curves are plotted for the purpose of comparison; the ultimate rates of mortality by the Government Annuity (1910) experience and the rates arrived at by Mr. Neil Campbell in the small investigation already referred to.

It will be seen that the new rates are throughout lighter than those by the Government Annuity Table but that otherwise the curves run very closely together. Another tribute is thus furnished to the extraordinarily light mortality experienced by female life tenants and at the same time some weight is given to the point raised in the last paragraph when the antiquity of the $O^{[af]}$ Table was under discussion.

Mr. Neil Campbell's Table.

Diagram 4 shows that up to age 77 the rates by Mr. Neil Campbell's Table agree fairly well with those of the present experience. At the older ages, however, there is a marked divergence and the rates according to the new table follow those of the Annuity experiences much more closely

than those of Mr. Campbell's investigation. It is I think reasonable to attribute this divergence to paucity of data in the latter investigation rather than to a definite change in the character of the mortality curve since 1902. In any event, the general conclusion drawn by Mr. Campbell as to the abnormally light mortality of life tenants is in no way vitiated.

In this connection the following table is of more than a little interest :

Age Group	Number of years Exposure to Risk. Campbell's Experience	Expected Deaths by New Table (Females)	Actual Deaths. Campbell's Experience
55–59	176	2·37	2·
60–64	296	5·68	6·
65–69	407	11·78	13·
70–74	422	18·69	23·
75–79	342	23·94	23·
80–84	214	23·49	24·
85–89	94	16·37	13·
...	1,951	102·32	104·

It will be noticed that not only is the mortality lighter in general under the new table but that the improvement is all at the younger ages ; this is interesting in view of the difference between the dates at which the respective investigations were made.

MALES.

(17). The mortality among the male life tenants proved to be surprisingly heavy, as a mere glance at the values in Table B will show. The marked difference between the male and female rates is shown by Diagram 5.

Following the plan adopted in the case of the females a comparison is made in Diagram 6 between the values of q_x by the Life Tenants (Male) Table, the Carlisle Table, the $O^{[am]}$ (ult.) Table and the Government Annuitants (1910) (ult.) Table ; the result is rather striking. It will be seen that up to about age 79 the rates according to all four tables run closely together but that thereafter the rates of mortality experienced among life tenants are very heavy and are not even approximately represented by either of the annuitant curves.

Unfortunately the data of the Life Tenants (Male) Table at the old ages were too few for a really reliable conclusion to be drawn in this respect.

Diagram 1.

Rates of Mortality. Females.

Ungraduated ×—×—×—×

Graduated ——————

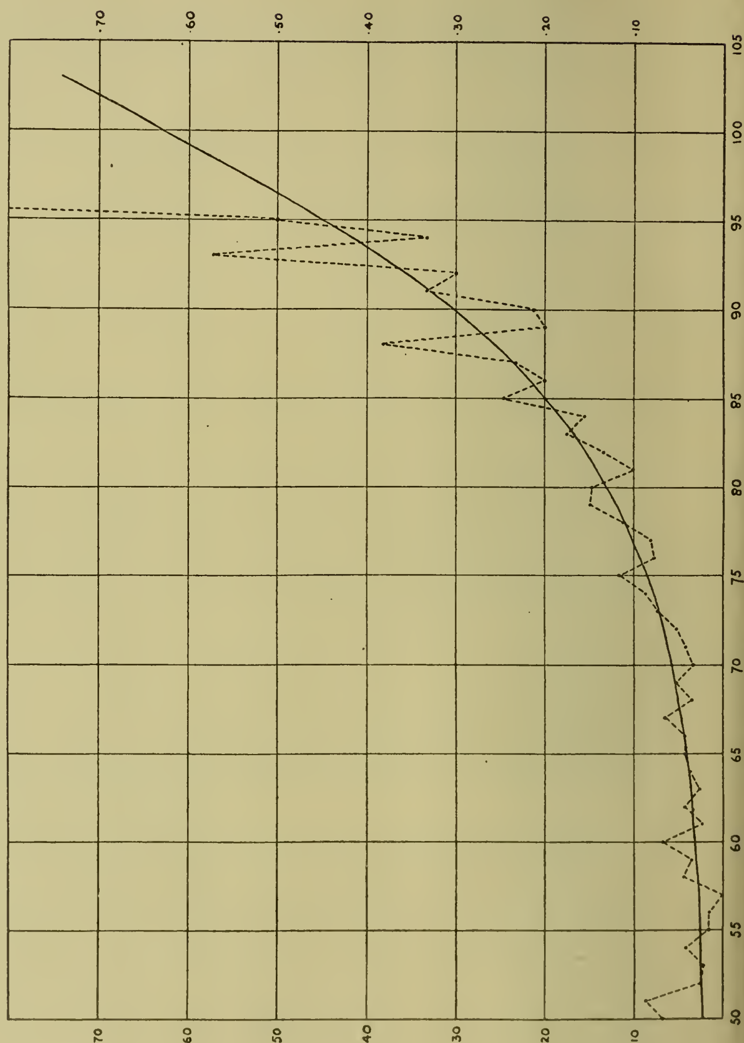


Diagram 2.
Rates of Mortality Males.
Ungraduated -·-·-·-·-·
Graduated ———

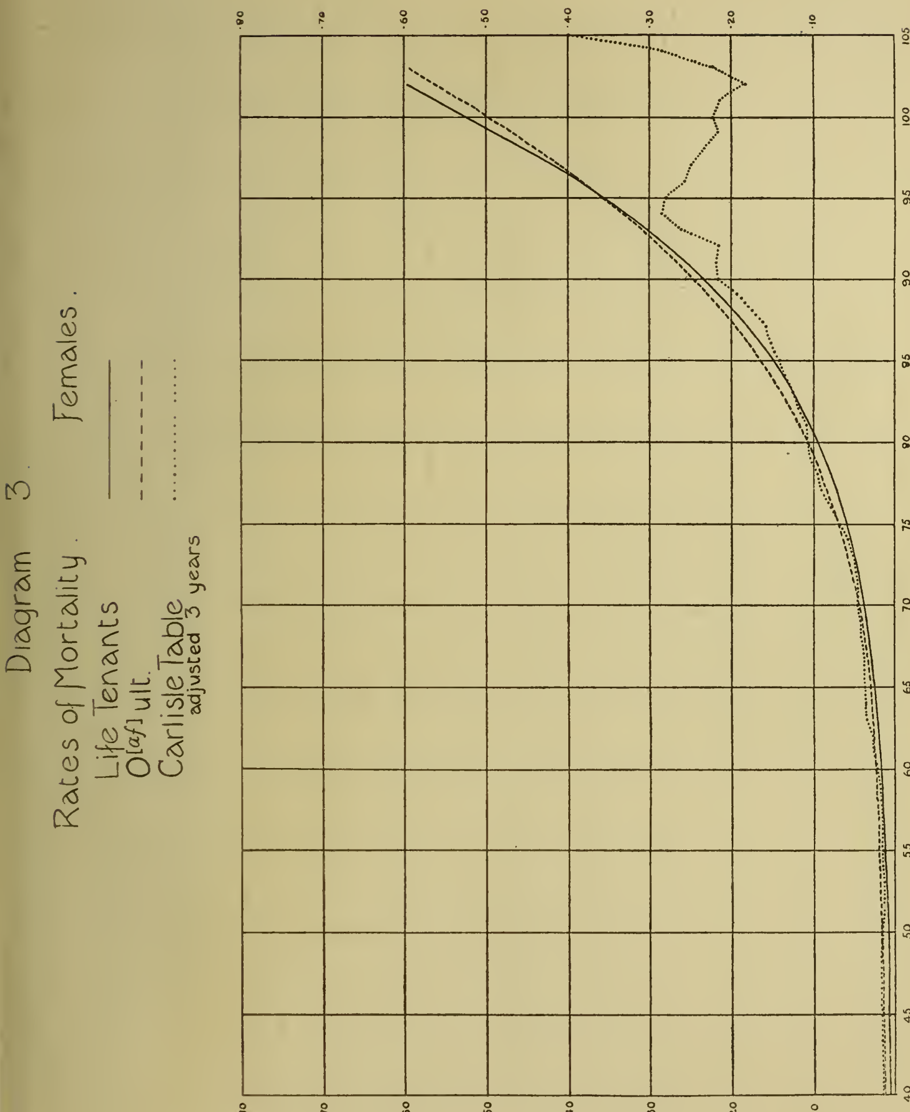


Diagram 3.
Rates of Mortality. Females.
Life Tenants
O[af] ult.
Carlisle Table adjusted 3 years

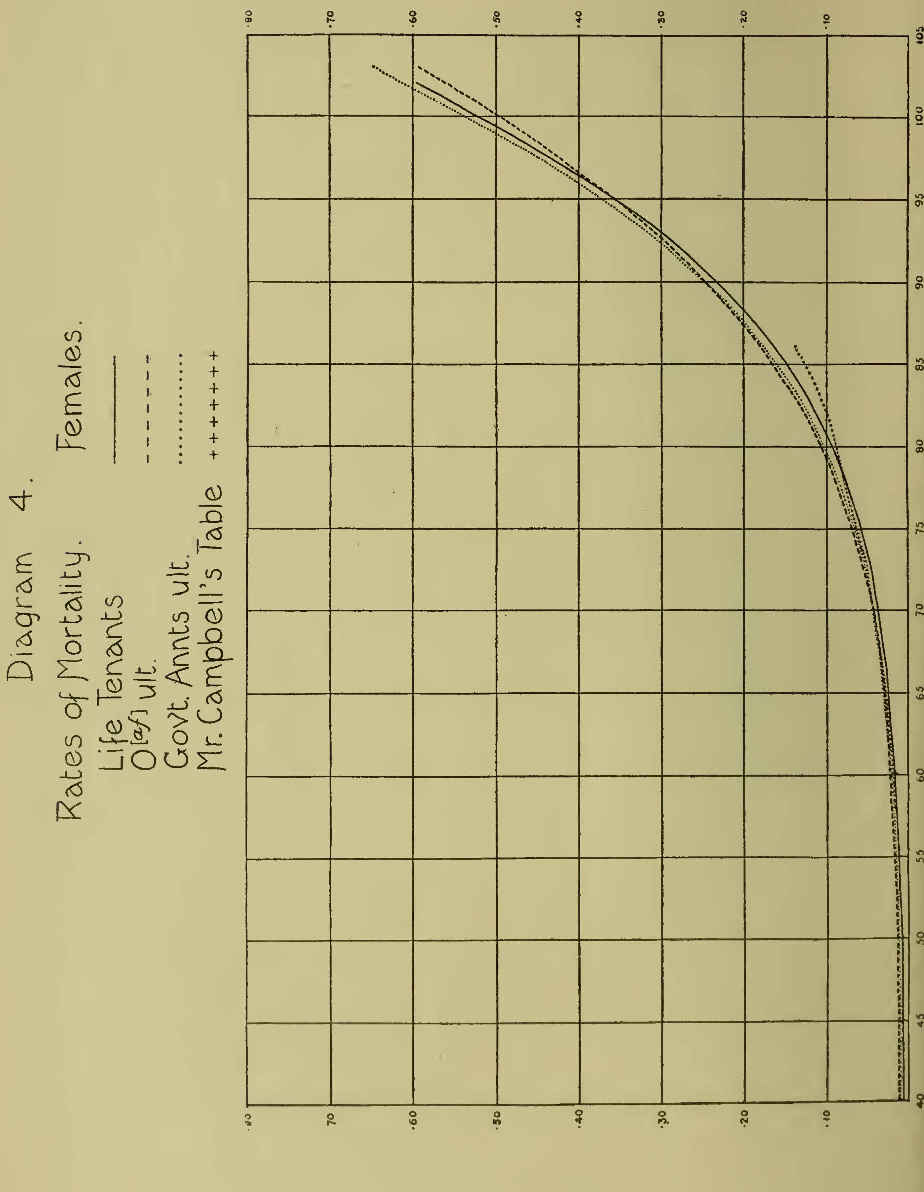


Diagram 4.
Rates of Mortality. Females.
Life Tenants
O[af] ult.
Govt. Annts ult.
Mr. Campbell's Table
·10
·20
·30
·40
·50
·60
·70
·80
40
45
50
55
60
65
70
75
80
85
90
95
100
105

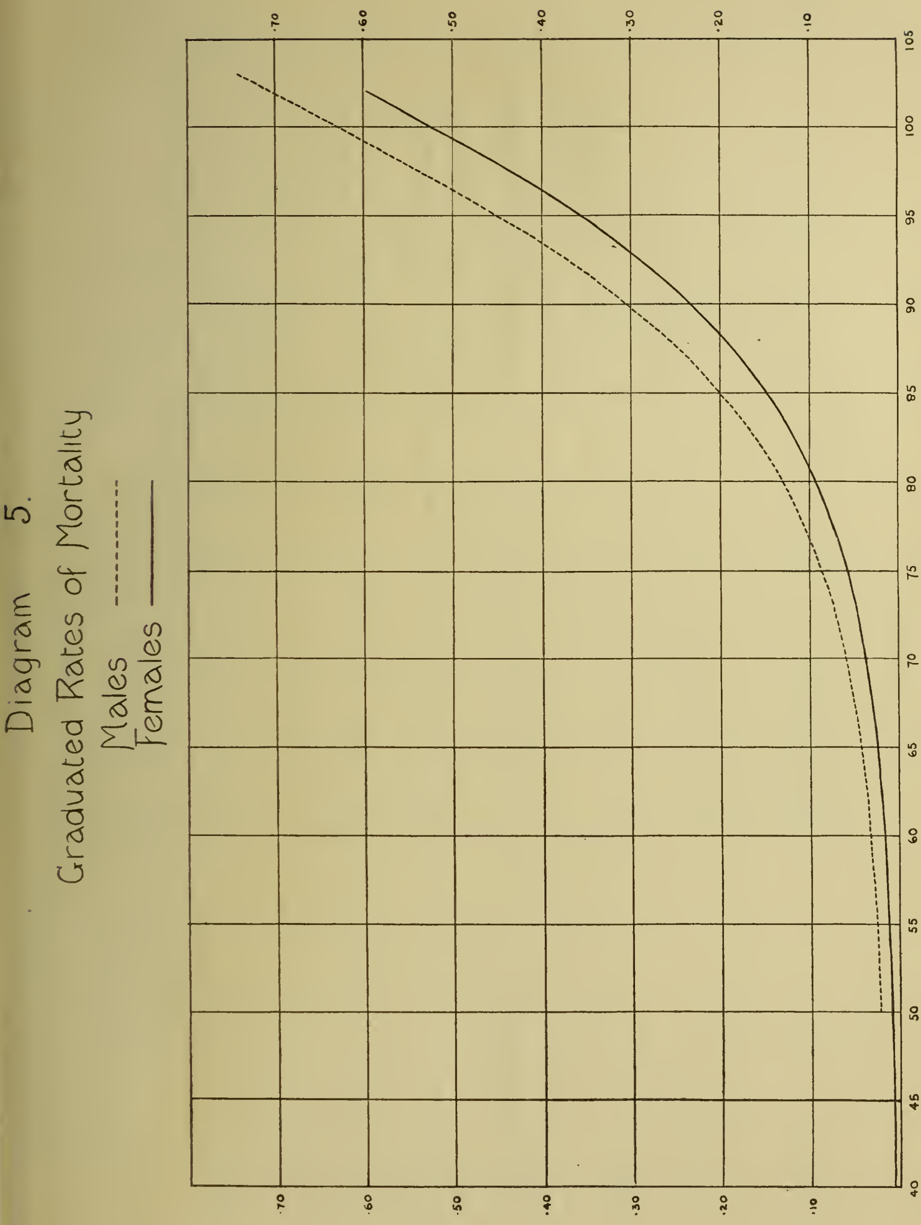


Diagram 5.
Graduated Rates of Mortality
Males
Females
.70
.60
.50
.40
.30
.20
.10
40
45
50
55
60
65
70
75
80
85
90
95
100
105

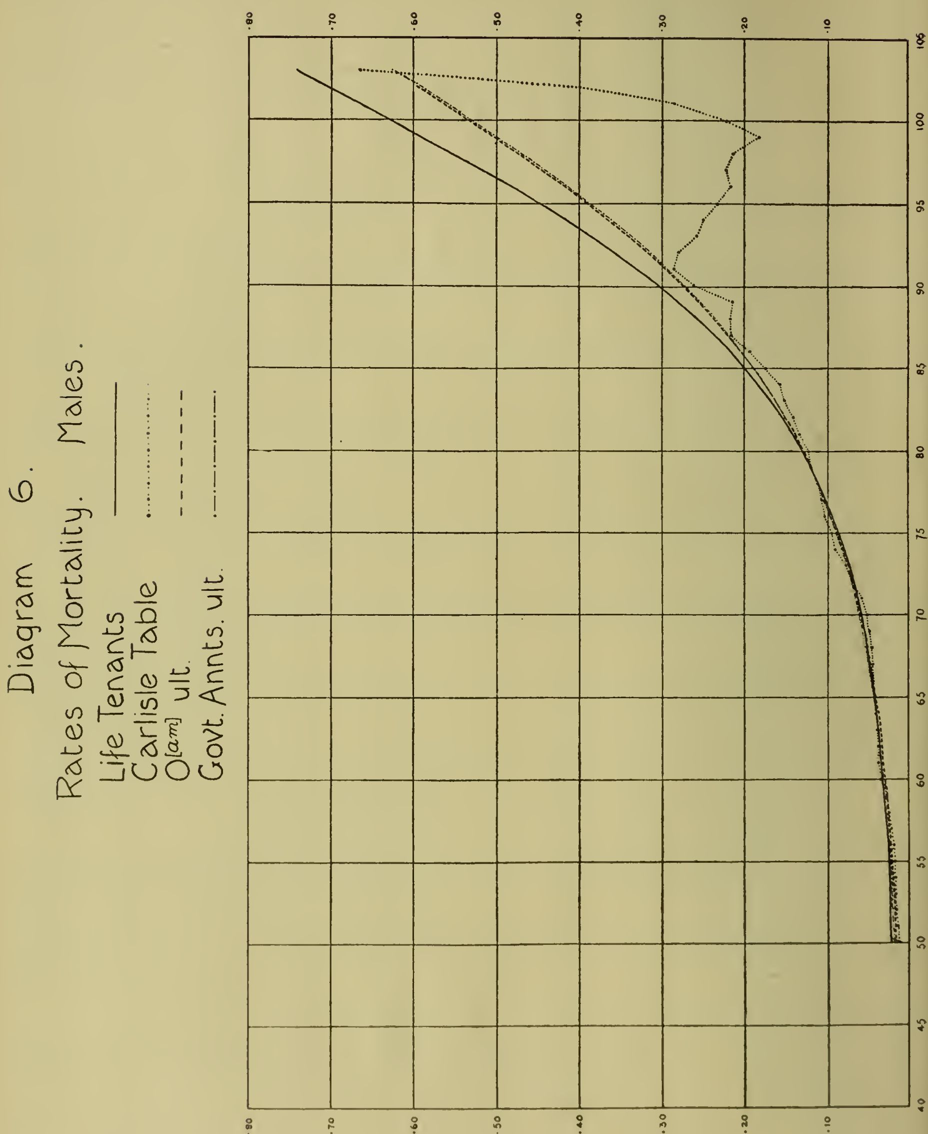


Diagram 6.
Rates of Mortality. Males.
Life Tenants
Carlisle Table
O[am] ult.
Govt. Annts. ult.
.80
.70
.60
.50
.40
.30
.20
.10
40
45
50
55
60
65
70
75
80
85
90
95
100
105

The entire unsuitability of the Carlisle Table for employment in the valuation of reversions is, however, again demonstrated and the general arguments used in this connection in the case of the female rates apply with equal force to the male rates.

Some Further Comparisons.

Expectations of Life. A preliminary idea of the financial effect of adopting the new table in the future for the valuation of reversions can be gleaned from the following comparative table of expectations of life.

In the case of the $O^{[af]}$ and Government Annuitants (1910) experiences, the select values, $e_{[x]}$, have been tabulated, since the general practice in the past has been to use the select single premiums in valuation.

Females.—Values of e_x.

Age	Life Tenants (1)	Carlisle (rated down 3 years) (2)	$O^{[af]}$ (3)	Government Annuity (1910) Select (4)	Δ (1)–(2) (5)	Δ (1)–(3) (6)	Δ (1)–(4) (7)
40	32·90	29·14	29·66	30·53	3·76	3·24	2·37
45	28·92	25·84	26·27	26·98	3·08	2·65	1·94
50	24·99	22·67	22·99	23·44	2·32	2·00	1·55
55	21·14	19·18	19·75	19·96	1·96	1·39	1·18
60	17·46	15·71	16·51	16·62	1·75	·95	·84
65	14·00	12·81	13·32	13·49	1·19	·68	·51
70	10·86	10·25	10·38	10·64	·61	·48	·22
75	8·11	7·66	7·86	8·15	·45	·23	–·04
80	5·79	5·90	5·81	6·06	–·11	–·02	–·27

Males—Values of e_x.

Age	Life Tenants (1)	Carlisle (2)	$O^{[am]}$ (3)	Government Annuity (1910) Select (4)	Δ (1)–(2) (5)	Δ (1)–(3) (6)	Δ (1)–(4) (7)
50	19·40	20·61	20·73	20·46	–1·21	–1·33	–1·06
55	16·56	17·08	17·41	17·24	– ·52	– ·85	– ·68
60	13·75	13·84	14·32	14·23	– ·09	– ·57	– ·48
65	11·06	11·29	11·52	11·48	– ·23	– ·46	– ·42
70	8·56	8·68	9·04	9·05	– ·12	– ·48	– ·49
75	6·35	6·51	6·92	6·96	– ·16	– ·57	– ·61
80	4·48	5·01	5·16	5·23	– ·53	– ·68	– ·75

The extremely light mortality among female life tenants is emphasized by columns (6) and (7) which show that, for the great majority of ages, the expectations of life exceed even the select values according to the annuity tables.

The corresponding comparison for male lives is instructive, and presents, from a financial point of view, a much more cheerful appearance.

Although for the most part the differences are small it will be seen that in the case of male lives no loss has been sustained in the past on account of any difference between the mortality actually experienced and that upon which the valuation factor was based.

Actual and expected deaths —Carlisle mortality.

Further light on the error involved in the use of the Carlisle Table is obtained by multiplying the numbers exposed to risk at each age according to the present investigation (females) by the Carlisle values of q_x rated down three years, and comparing the resulting number of expected deaths with those that actually occurred. The number of expected deaths thus ascertained is 1,435, while the actual deaths were 1,282. On the basis of the particular age distribution concerned, therefore, the use of the Carlisle Table even with the customary adjustment involved an overestimate of the mortality of 12 per-cent.

The mortality of life tenants under contingent reversions.

The numbers exposed to risk were separately ascertained, in full aggregate form, for contingent reversions, and the expected deaths calculated according to the rates for the whole experience. The following resulted :

Contingent Reversions.—Females.

Age Group	Number of years of Exposure to Risk	Expected Deaths (Whole Experience)	Actual Deaths
–55	137	1·36	1
56–60	230	3·30	4
61–65	333	6·93	7
66–70	454	14·27	14
71–75	470	22·82	18
76–80	449	34·63	30
81–85	286	34·50	39
86–90	141	26·46	23
91–	39	11·50	20
...	2,539	155·77	156

In view of the fact that in the combined experience 90 per-cent of the data related to absolute reversions, it may reasonably be inferred from the above that the general character of the mortality is unaffected by the type of reversion involved.

Monetary Values.

(19). In order to make the paper of some practical use as well as of scientific interest, I have calculated certain monetary functions and these will be found in the Tables appended to the paper.

Females. Values of N_x, D_x, a_x and A_x, at 5 per-cent, 6 per-cent and 7 per-cent based upon the female rates of mortality are given in Tables C and D.

Although these functions by no means form a complete equipment for the actuary in his dealings with reversions, yet they are at any rate sufficient for the valuation of simple reversions on single lives, which, after all, constitute the bulk of the cases met with in practice. The range of rates of interest should satisfy requirements for some time to come.

To facilitate the calculation of single premiums or annuity values at other rates of interest, should these be required, values are also given in Table F of the term-certain corresponding to each value of A_x, *i.e.*, values of n such that $v^n = A_x$. If these are employed in the manner indicated in my note in *J.I.A.*, vol. lii, p. 171, and outlined in Appendix A to this paper, it will be found that sufficiently accurate values of A_x or a_x for any other rate of interest within reasonable limits can be obtained with great rapidity.

I had hoped to be able to give at least one table of joint life annuities, equal ages, on the basis of the female mortality, but pressure of other duties has prevented this.

Males. The conflict between business instinct and the craving for scientific accuracy, together with the ever present necessity of adopting a cautious attitude, will probably result in but little practical use being made of the male table ; in consequence I only give one table of monetary functions based on male mortality, and this mainly for comparative purposes. In Table E will be found values of N_x, D_x, a_x and A_x at 6 per-cent.

The use of the New Tables in Future—Some General Considerations.

(20). For reasons which are too obvious to require enumeration it will not, I think, be disputed that the new table furnishes a

more accurate basis for the valuation of reversions in future, than any of the tables that one has been compelled to use in the past.

It is nevertheless well to remember that however accurately it may reflect the mortality that has actually been experienced among life tenants during the last fifteen years or so, its value as an instrument for determining the future course of mortality is affected by influences which may introduce considerable error into the prediction, but which, from their very nature, cannot be measured.

One of these influences, the passing of time, which brings in its train improvements in medical science and sanitation, advance in education and the like, and in consequence a general improvement in the vitality of all classes, has already been commented upon. Another influence of a similar character, but of more particular applicability in the present instance, is the effect of the strain of the war on the health and longevity of all those who lived through the war period. Whether, in the case of the special class of lives under consideration, this effect will be little or much it is difficult to forecast; if it is found to exist at all it will in any event counteract, if only for a short period, the general improvement in vitality that might otherwise be expected.

A further factor not to be lost sight of, also a direct result of the war, is the change that has taken place in the general social structure and more particularly in the distribution of wealth during recent years. Without enlarging upon the point it is easy to see that this may result in a material alteration in the class of life forming the general body of life-tenants, and this in its turn will be reflected in the rates of mortality.

The combined effect of these influences is impossible of determination *a priori*. Fortunately, the close of the present investigation coincides roughly with the change in conditions, and it must be left to a future investigation to provide definite information upon what at present can only be a matter of conjecture.

CONCLUSION.

(21). It will now be convenient and useful to bring together in summary form the main facts that have been established by the present investigation and the more important conclusions to be drawn therefrom. These are:

(*a*) that there is no evidence that vendors of reversions exercise selection against purchasers to a significant extent and in consequence the mortality of life tenants can be sufficiently represented by a full aggregate table.

(*b*) that the mortality experienced by female life tenants has been in general extremely light, whereas the mortality among the males has been comparatively heavy, especially at ages above 80.

Bearing in mind the bases that have been adopted for the valuation of reversions in the past, it follows that on the whole, purchasers have been giving too high a price for reversions on female lives, compensated in part by a very slight gain in respect of male lives.

Speaking very broadly and on the assumption that valuations were all made on the basis of the Carlisle Table, with the customary adjustment for female lives, the loss in respect of females has been equivalent to an average depreciation per reversion of about 5 per-cent in the fund and the gain in respect of males equivalent to an appreciation of about 1 per-cent.

(*c*) that the Carlisle Table is unsuitable for employment in the valuation of reversions, especially when the life tenant is a female, since the general character of the Carlisle curve is quite different from that representing the mortality of life tenants.

(*d*) that tables based on the mortality experienced among purchasers of life annuities are also open to objection.

(1) because of the presence of the force of selection as well as the force of mortality.

(2) because those so far published exhibit in general too heavy a rate of mortality.

(*e*) that so far as can be judged from the data available the mortality among life tenants in cases where an option to repay is given is appreciably heavier than among the main body of lives.

(*f*) that in choosing a mortality table for valuation purposes it is not necessary to take into consideration the type of reversion involved.

The above items call for mention on account of their novelty or special importance ; to extend the list would be unnecessary recapitulation.

It would be idle to pretend that the treatment of the data

has been so complete that everything of interest or value has been extracted therefrom. All I venture to claim is that I have extracted enough to justify my presenting the results to this Institute to-night, and in any case the data are always available for further analysis if such be deemed necessary or desirable.

In conclusion I should like to express my sincere thanks to Messrs. F. C. Atkins, F.I.A. and C. A. Penny, A.I.A., for much valuable assistance and to the many others who have so readily helped with the numerical work.

APPENDIX A.

Method of employing Table F for obtaining values of A_x of a_x at rates of interest other than those for which these functions are tabulated.

The method is based on the assumption that with i as the only variable, the value of n which satisfies the equation $a_{\overline{n}|}=a_x$ is a linear function and can thus be made the subject of first difference interpolation; that this assumption is sufficiently near the truth to be employed in most cases met with in practice, has been demonstrated in the *Journal* (vol. lii, p. 171).

In view of the relation between annuity and assurance functions it is clear that if the assumption is sufficiently accurate in the case of the former it must also be sufficiently accurate in the case of the latter, *i.e.*, the value of n such that $v^n=A_x$ must also be a linear function if i is the only variable. The only point to notice is that for any given age the equivalent term certain is one year greater in the case of A_x than in the case of a_x.

In the present instance the values of the terms certain equivalent to the known values of A_x have been tabulated since this function is the one most likely to be required in practice. To facilitate calculations the differences are also given, the symbols $\Delta n^{5\%}$ and $\Delta n^{6\%}$ being employed to denote the change in value of the term certain when the rate of interest is increased by 1 per-cent.

Two examples will suffice to illustrate the method of employment of the table:

(*a*) Required A_{64} at $5\frac{3}{4}$ per-cent.

$$n^{5\frac{3}{4}\%} = n^{6\%} - \frac{1}{4}\Delta n^{5\%}$$
$$= 13{\cdot}858 + {\cdot}072 = 13{\cdot}930$$

Whence value required $= v^{13\cdot930}_{5\frac{3}{4}\%} = {\cdot}45901$ from annuity-certain tables. It is sufficient to use first difference interpolation for the fractional portion of n.

(*b*) Required A_{50} at $7\frac{1}{2}$ per-cent.

$$n^{7\frac{1}{2}\%} = n^{7\%} + \frac{1}{2}\Delta n^{6\%}$$
$$= 21{\cdot}733 - {\cdot}295 = 21{\cdot}438$$

Whence value required $= v^{21\cdot438}_{7\frac{1}{2}\%} = {\cdot}21230$.

If annuity-values are required these can either be obtained from the assurance values by means of conversion tables or direct from annuity-certain tables by using a term one year less than that found for the assurance functions.

It will be seen that a tabulation of equivalent terms-certain is a useful adjunct to the monetary functions of a mortality table, and this might be worth bearing in mind on future occasions.

APPENDIX B.

Females.—Analysis of Deaths during first 10 *years.*
Expected Deaths by Ultimate (*ex.* 10 *years*) *Table*

Year of Duration	Deaths		$100\frac{(2)}{(3)}$	Quinquennium of Duration	Deaths		$100\frac{(6)}{(7)}$
	Actual	Expected			Actual	Expected	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0–1	44	57·08	77·1				
1–2	63	59·97	105·1				
2–3	54	59·14	91·4	1	282	301·36	93·6
3–4	54	61·99	87·1				
4–5	67	63·18	94·3				
5–6	54	60·99	88·5				
6–7	59	61·55	95·9				
7–8	60	62·30	96·3	2	280	306·68	91·3
8–9	63	60·73	103·8				
9–10	44	61·11	72·0				
...	562	608·04	92·4	...	562	608·04	92·4

TABLE C.

Mortality of Life Tenants. Females. Values of D_x *and* N_x.

Age	5 %		6 %		7 %	
	D_x	N_x	D_x	N_x	D_x	N_x
40	134,347	2,011,355	91,953	1,221,072	63,161	751,144
41	127,162	1,884,193	86,215	1,134,857	58,666	692,478
42	120,343	1,763,850	80,821	1,054,036	54,482	637,996
43	113,868	1,649,982	75,752	978,284	50,587	587,409
44	107,721	1,542,261	70,986	907,298	46,961	540,448
45	101,882	1,440,379	66,505	840,793	43,586	496,862
46	96,337	1,344,042	62,292	778,501	40,443	456,419
47	91,069	1,252,973	58,331	720,170	37,517	418,902
48	86,064	1,166,909	54,605	665,565	34,793	384,109
49	81,307	1,085,602	51,100	614,465	32,255	351,854
50	76,785	1,008,817	47,802	566,663	29,892	321,962
51	72,485	936,332	44,700	521,963	27,690	294,272
52	68,394	867,938	41,779	480,184	25,639	268,633
53	64,502	803,436	39,030	441,154	23,728	244,905
54	60,798	742,638	36,442	404,712	21,948	222,957
55	57,271	685,367	34,004	370,708	20,288	202,669
56	53,912	631,455	31,708	339,000	18,741	183,928
57	50,712	580,743	29,544	309,456	17,299	166,629
58	47,662	533,081	27,505	281,951	15,955	150,674
59	44,753	488,328	25,583	256,368	14,701	135,973
60	41,978	446,350	23,770	232,598	13,532	122,441
61	39,330	407,020	22,061	210,537	12,441	110,000
62	36,803	370,217	20,448	190,089	11,424	98,576
63	34,389	335,828	18,927	171,162	10,475	88,101
64	32,083	303,745	17,491	153,671	9,590	78,511
65	29,879	273,866	16,132	137,539	8,765	69,746
66	27,774	246,092	14,854	122,685	7,995	61,751
67	25,762	220,330	13,648	109,037	7,277	54,474
68	23,839	196,491	12,510	96,527	6,608	47,866
69	22,001	174,490	11,437	85,090	5,984	41,882
70	20,246	154,244	10,425	74,665	5,404	36,478
71	18,571	135,673	9,473	65,192	4,864	31,614
72	16,973	118,700	8,576	56,616	4,363	27,251
73	15,452	103,248	7,734	48,882	3,897	23,354
74	14,005	89,243	6,943	41,939	3,466	19,888
75	12,631	76,612	6,203	35,736	3,068	16,820
76	11,330	65,282	5,512	30,224	2,701	14,119
77	10,102	55,170	4,868	25,356	2,363	11,756
78	8,948	46,232	4,271	21,085	2,054	9,702
79	7,866	38,366	3,719	17,366	1,772	7,930
80	6,859	31,507	3,213	14,153	1,516	6,414
81	5,927	25,580	2,750	11,403	1,285	5,129
82	5,069	20,511	2,330	9,073	1,079	4,050
83	4,288	16,223	1,952	7,121	896	3,154
84	3,582	12,641	1,615	5,506	734	2,420
85	2,951	9,690	1,318	4,188	594	1,826
86	2,394	7,296	1,059	3,129	473	1,353
87	1,910	5,386	837	2,292	370	983
88	1,495	3,891	649	1,643	284	699
89	1,146	2,745	493	1,150	214	485
90	859	1,886	366	784	157	328
91	627	1,259	265	519	113	215
92	445	814	186	333	78·4	137
93	306	508	127	206	52·9	84·3
94	203	305	83·3	123	34·5	49·8
95	130	175	52·7	70·0	21·6	28·2
96	79·3	96·16	32·0	38·0	13·0	15·2
97	46·2	49·96	18·4	19·6	7·41	7·75
98	25·5	24·46	10·1	9·51	4·02	3·73
99	13·3	11·16	5·21	4·30	2·06	1·67
100	6·50	4·66	2·52	1·78	·985	·688
101	2·95	1·71	1·13	·65	·439	·249
102	1·24	·47	·471	·179	·181	·068
103	·47	...	·179	...	·068	...

TABLE D.

Mortality of Life Tenants. Females.

Age	5 %		6 %		7 %	
	a_x	A_x	a_x	A_x	a_x	A_x
40	14·971	·23947	13·279	·19175	11·893	·15653
1	14·817	·24681	13·163	·19832	11·804	·16236
2	14·657	·25443	13·042	·20517	11·710	·16850
3	14·490	·26238	12·914	·21241	11·612	·17492
4	14·317	·27062	12·781	·21994	11·508	·18172
5	14·138	·27914	12·643	·22775	11·400	·18879
6	13·951	·28805	12·498	·23597	11·285	·19631
7	13·759	·29719	12·346	·24457	11·166	·20409
8	13·559	·30671	12·189	·25345	11·040	·21234
9	13·352	·31657	12·025	·26274	10·909	·22091
50	13·138	·32676	11·854	·27241	10·771	·22993
1	12·918	·33724	11·677	·28243	10·627	·23935
2	12·690	·34810	11·493	·29285	10·477	·24917
3	12·456	·35923	11·303	·30360	10·321	·25937
4	12·215	·37071	11·106	·31475	10·159	·26997
5	11·967	·38253	10·902	·32631	9·990	·28103
6	11·713	·39462	10·691	·33824	9·814	·29254
7	11·452	·40704	10·474	·35052	9·632	·30445
8	11·185	·41976	10·251	·36315	9·444	·31675
9	10·912	·43276	10·021	·37617	9·249	·32950
60	10·633	·44605	9·785	·38953	9·048	·34265
1	10·349	·45957	9·543	·40323	8·841	·35620
2	10·059	·47338	9·296	·41721	8·629	·37007
3	9·766	·48733	9·043	·43153	8·410	·38439
4	9·467	·50157	8·786	·44608	8·187	·39898
5	9·166	·51590	8·526	·46079	7·958	·41396
6	8·861	·53043	8·259	·47591	7·724	·42927
7	8·553	·54510	7·990	·49113	7·486	·44484
8	8·242	·55990	7·716	·50664	7·244	·46067
9	7·931	·57471	7·440	·52226	6·998	·47677
70	7·618	·58962	7·162	·53800	6·750	·49299
1	7·306	·60447	6·882	·55385	6·499	·50941
2	6·993	·61938	6·601	·56975	6·246	·52596
3	6·682	·63419	6·321	·58560	5·992	·54258
4	6·372	·64895	6·040	·60151	5·737	·55926
5	6·065	·66357	5·761	·61730	5·482	·57594
6	5·762	·67800	5·484	·63298	5·228	·59256
7	5·462	·69228	5·208	·64861	4·975	·60911
8	5·167	·70634	4·937	·66394	4·724	·62553
9	4·877	·72015	4·669	·67911	4·475	·64182
80	4·594	·73362	4·405	·69406	4·230	·65785
1	4·316	·74685	4·147	·70866	3·989	·67362
2	4·046	·75971	3·894	·72298	3·753	·68906
3	3·783	·77224	3·648	·73691	3·521	·70423
4	3·529	·78433	3·409	·75043	3·295	·71902
5	3·284	·79600	3·177	·76356	3·076	·73335
6	3·048	·80724	2·953	·77625	2·863	·74728
7	2·820	·81810	2·737	·78847	2·658	·76069
8	2·602	·82847	2·529	·80024	2·460	·77364
9	2·394	·83838	2·330	·81151	2·269	·78614
90	2·196	·84781	2·140	·82226	2·087	·79805
1	2·008	·85676	1·960	·83245	1·913	·80943
2	1·830	·86524	1·788	·84219	1·747	·82029
3	1·661	·87328	1·626	·85136	1·591	·83050
4	1·503	·88081	1·472	·86008	1·443	·84018
5	1·354	·88789	1·328	·86823	1·302	·84940
6	1·214	·89457	1·192	·87593	1·170	·85804
7	1·083	·90081	1·064	·88317	1·046	·86615
8	·958	·90676	·943	·89012	·927	·87393
9	·839	·91243	·826	·89664	·813	·88139
100	·717	·91824	·708	·90332	·697	·88898
101	·580	·92476	·573	·91096	·565	·89762

TABLE E.

Mortality of Life Tenants.

6 PER-CENT. *Males.* 6 PER-CENT.

Age	D_x	N_x	a_x	A_x
30	174,110	2,095,868	12·038	·26200
1	161,433	1,934,435	11·983	·26511
2	149,668	1,784,767	11·925	·26840
3	138,749	1,646,018	11·863	·27191
4	128,614	1,517,404	11·798	·27559
5	119,207	1,398,197	11·729	·27949
6	110,476	1,287,721	11·656	·28362
7	102,371	1,185,350	11·579	·28798
8	94,847	1,090,503	11·498	·29257
9	87,863	1,002,640	11·411	·29749
40	81,379	921,261	11·321	·30258
1	75,359	845,902	11·225	·30802
2	69,769	776,133	11·124	·31373
3	64,579	711,554	11·018	·31974
4	59,759	651,795	10·907	·32602
5	55,283	596,512	10·790	·33264
6	51,126	545,386	10·668	·33955
7	47,264	498,122	10·539	·34685
8	43,677	454,445	10·405	·35444
9	40,344	414,101	10·264	·36241
50	37,247	376,854	10·118	·37068
1	34,370	342,484	9·965	·37934
2	31,696	310,788	9·806	·38834
3	29,210	281,578	9·640	·39774
4	26,899	254,679	9·468	·40747
5	24,751	229,928	9·290	·41755
6	22,754	207,174	9·105	·42802
7	20,897	186,277	8·914	·43883
8	19,170	167,107	8·717	·44998
9	17,564	149,543	8·514	·46147
60	16,070	133,473	8·306	·47324
1	14,681	118,792	8·091	·48541
2	13,390	105,402	7·872	·49781
3	12,189	93,213	7·647	·51054
4	11,073	82,140	7·418	·52351
5	10,036	72,104	7·185	·53670
6	9,072	63,032	6·948	·55012
7	8,178	54,854	6·708	·56370
8	7,349	47,505	6·464	·57751
9	6,581	40,924	6·219	·59138
70	5,870	35,054	5·972	·60536
1	5,214	29,840	5·723	·61945
2	4,609	25,231	5·475	·63349
3	4,053	21,178	5·226	·64758
4	3,543	17,635	4·978	·66163
5	3,077	14,558	4·732	·67555
6	2,653	11,905	4·488	·68936
7	2,269	9,636	4·247	·70300
8	1,924	7,712	4·009	·71647
9	1,615	6,097	3·776	·72966

TABLE E—*continued.*

Mortality of Life Tenants.

6 PER-CENT. *Males.* 6 PER-CENT.

Age	D_x	N_x	a_x	A_x
80	1,341	4,756	3·547	·74262
1	1,100	3,655	3·323	·75530
2	891	2,764	3·105	·76764
3	710	2,054	2·894	·77958
4	557	1,497	2·689	·79153
5	429	1,068	2·492	·80234
6	324	744	2·302	·81310
7	239	505	2·120	·82340
8	172	333	1·946	·83324
9	120	213	1·780	·84264
90	81·7	132	1·662	·84932
1	53·6	78·8	1·473	·86002
2	33·8	45·0	1·332	·86800
3	20·5	24·5	1·200	·87547
4	11·8	12·7	1·076	·88249
5	6·50	6·25	·960	·88905
6	3·37	2·88	·853	·89511
7	1·64	1·24	·753	·90077
8	·743	·492	·660	·90604
9	·312	·180	·575	·91085
100	·120	·060	·497	·91526
1	·042	·018	·423	·91945
2	·013	·005	·351	·92353
3	·005	...	...	...

TABLE F.

Mortality of Life Tenants. Females. Full Aggregate.
Equivalent Terms Certain for obtaining Assurance Values.

Age	VALUES OF n SUCH THAT $v^n = A_x$				
	5 %	6 %	7 %	$-\Delta 5$ %	$-\Delta 6$ %
40	29·300	28·350	27·418	·950	·932
1	28·682	27·771	26·874	·911	·897
2	28·055	27·187	26·328	·868	·859
3	27·429	26·594	25·774	·835	·820
4	26·793	25·990	25·210	·803	·780
5	26·157	25·398	24·648	·759	·750
6	25·516	24·788	24·065	·728	·723
7	24·872	24·173	23·497	·699	·676
8	24·227	23·563	22·906	·664	·657
9	23·581	22·940	22·326	·641	·614
50	22·927	22·324	21·733	·603	·591
1	22·283	21·704	21·139	·579	·565
2	21·635	21·078	20·547	·557	·531
3	20·984	20·465	19·947	·519	·518
4	20·344	19·842	19·361	·502	·481
5	19·701	19·225	18·766	·476	·459
6	19·059	18·610	18·171	·449	·439
7	18·428	17·991	17·585	·437	·406
8	17·796	17·391	16·992	·405	·399
9	17·170	16·784	16·417	·386	·367
60	16·553	16·185	15·835	·368	·350
1	15·936	15·594	15·264	·342	·330
2	15·333	15·002	14·700	·331	·302
3	14·738	14·430	14·135	·308	·295
4	14·145	13·858	13·589	·287	·269
5	13·571	13·303	13·037	·268	·266
6	12·996	12·749	12·507	·247	·242
7	12·443	12·206	11·973	·237	·233
8	11·890	11·676	11·464	·214	·212
9	11·358	11·152	10·950	·206	·202
70	10·831	10·645	10·462	·186	·183
1	10·323	10·144	9·970	·179	·174
2	9·822	9·661	9·505	·161	·156
3	9·339	9·188	9·038	·151	·150
4	8·865	8·729	8·597	·136	·132
5	8·412	8·285	8·159	·127	·126
6	7·966	7·852	7·741	·114	·111
7	7·544	7·437	7·335	·107	·102
8	7·128	7·030	6·936	·098	·094
9	6·734	6·648	6·562	·086	·086
80	6·354	6·273	6·195	·081	·078
1	5·983	5·913	5·844	·070	·069
2	5·638	5·574	5·513	·064	·061
3	5·303	5·245	5·188	·058	·057
4	4·979	4·929	4·879	·050	·050
5	4·682	4·636	4·592	·046	·044
6	4·395	4·353	4·313	·042	·040
7	4·118	4·081	4·044	·037	·037
8	3·860	3·829	3·799	·031	·030
9	3·619	3·591	3·565	·028	·026
90	3·390	3·365	3·342	·025	·023
1	3·172	3·151	3·129	·021	·022
2	2·967	2·949	2·930	·018	·019
3	2·781	2·767	2·752	·014	·015
4	2·607	2·594	2·582	·013	·012
5	2·443	2·432	2·421	·011	·011

TABLE A.

Mortality of Life Tenants. Unadjusted Data.

Age	Females: Full Aggregate: Exposed to Risk	Deaths	q_x	Females: Truncated Aggregate excluding first FIVE Years: Exposed to Risk	Deaths	q_x	Females: Truncated Aggregate excluding first TEN Years: Exposed to Risk	Deaths	q_x	Males: Full Aggregate: Exposed to Risk	Deaths	q_x	Age
30	1	...	...	1	...	...	...	...	...	4	...	...	30
1	2	...	...	1	...	...	...	...	...	2	...	...	1
1	563	60	·1066	450	48	·1067	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	511	46	·0900	408	38	·0931	306	28	·0915	96	13	·1354	2
3	456	67	·1469	378	58	·1534	288	40	·1389	85	15	·1765	3
4	385	51	·1325	315	41	·1302	249	26	·1044	70	11	·1571	4
5	334	46	·1377	281	42	·1495	223	34	·1525	65	16	·2462	5
6	300	34	·1133	254	28	·1102	203	18	·0887	55	11	·2000	6
7	266	45	·1692	234	38	·1624	187	29	·1551	43	10	·2326	7
8	212	39	·1840	189	36	·1904	153	28	·1830	34	13	·3824	8
9	172	34	·1977	153	32	·2092	127	30	·2362	20	4	·2000	9
90	134	42	·3134	113	36	·3186	91	30	·3297	14	3	·2143	90
1	92	27	·2935	79	22	·2785	65	19	·2923	15	5	·3333	1
2	65	18	·2769	57	15	·2632	50	11	·2200	10	3	·3000	2
3	46	18	·3913	41	16	·3902	37	14	·3784	7	4	·5714	3
4	28	6	·2143	24	5	·2083	22	5	·2273	3	1	·3333	4
5	24	11	·4583	21	10	·4762	17	7	·4118	2	1	·5000	5
6	14	2	·1429	11	1	·0909	10	1	·1000	1	1	1·0000	6
7	12	7	·5833	10	7	·7000	9	7	·7778	...	...	...	7
8	3	...	...	2	...	...	2	...	...	...	...	...	8
9	3	3	1·	2	2	1·	2	2	1·	...	...	...	9
100	1	1	1·	1	1	1·	1	1	1·	...	...	...	100
1	1	...	...	...	...	...	...	...	...	...	...	...	1
2	1	1	1·	1	1	1·	...	...	...	...	...	...	2
...	23,226	1,282	...	16,043	1,000	...	9,666	720	...	4,622	363	...	...

TABLE A.

Mortality of Life Tenants. Unadjusted Data.

Age	Females									Males			Age
	Full Aggregate			Truncated Aggregate excluding first FIVE Years			Truncated Aggregate excluding first TEN Years			Full Aggregate			
	Exposed to Risk	Deaths	q_x	Exposed to Risk	Deaths	q_x	Exposed to Risk	Deaths	q_x	Exposed to Risk	Deaths	q_x	
30	1	...	...	1	...	...	...	...	...	4	...	...	30
1	2	...	...	1	...	...	...	...	...	2	...	...	1
2	4	...	...	2	...	...	1	...	...	3	...	...	2
3	4	...	...	3	...	...	1	...	...	...	...	...	3
4	4	...	...	3	...	...	1	...	...	...	...	...	4
5	5	...	...	4	...	...	1	...	...	...	...	...	5
6	7	...	...	4	...	...	1	...	...	4	...	...	6
7	9	...	...	5	...	...	2	...	...	5	...	...	7
8	12	...	...	6	...	...	3	...	...	7	...	...	8
9	17	...	...	7	...	...	5	...	...	6	...	...	9
40	24	...	...	9	...	...	5	...	...	7	...	...	40
1	28	1	·0357	13	1	...	5	...	...	...	...	...	1
2	36	1	·0278	16	...	...	7	...	...	6	...	...	2
3	46	...	...	20	...	...	9	...	...	9	...	...	3
4	55	...	...	24	...	...	10	...	...	2	...	...	4
5	69	...	...	29	...	...	11	...	...	...	...	...	5
6	77	1	·0130	32	1	...	14	1	·0714	11	...	...	6
7	87	2	·0230	38	2	...	15	2	·1333	15	...	...	7
8	110	...	...	42	...	...	17	...	...	5	...	...	8
9	132	5	·0379	51	4	...	21	3	·1429	23	...	...	9
50	150	...	...	61	...	...	20	...	...	29	2	·0690	50
1	170	3	·0176	70	2	·0286	21	...	...	34	3	·0882	1
2	197	1	·0051	85	...	...	25	...	...	38	1	·0263	2
3	232	1	·0043	102	...	...	29	...	...	45	1	·0222	3
4	271	4	·0148	130	...	...	39	...	...	48	2	·0417	4
5	313	11	·0351	154	7	·0454	51	3	·0588	58	1	·0172	5
6	358	7	·0196	174	4	·0230	60	2	·0333	61	1	·0164	6
7	399	3	·0075	185	...	...	76	...	...	79	...	...	7
8	452	9	·0199	231	2	·0087	98	1	·0102	89	4	·0449	8
9	510	4	·0078	265	2	·0076	116	...	...	108	4	·0370	9
60	559	7	·0125	298	3	·0101	124	2	·0161	115	8	·0696	60
1	620	8	·0129	350	4	·0114	149	2	·0134	132	3	·0227	1
2	659	11	·0167	397	8	·0202	163	4	·0245	135	6	·0444	2
3	710	12	·0169	449	10	·0223	202	4	·0198	141	4	·0284	3
4	752	16	·0213	489	11	·0225	229	2	·0087	155	6	·0387	4
5	787	15	·0191	529	6	·0113	251	3	·0120	165	7	·0424	5
6	819	14	·0171	560	9	·0161	284	3	·0106	163	7	·0429	6
7	857	23	·0268	589	16	·0272	325	7	·0215	164	11	·0670	7
8	877	33	·0376	602	21	·0349	359	14	·0390	167	6	·0359	8
9	863	28	·0324	599	19	·0317	359	15	·0418	170	9	·0529	9
70	847	22	·0260	613	18	·0294	377	12	·0318	169	6	·0355	70
1	850	45	·0529	637	39	·0612	392	24	·0612	167	7	·0419	1
2	843	41	·0486	638	32	·0502	398	19	·0477	174	9	·0517	2
3	813	45	·0553	616	33	·0536	389	27	·0694	174	13	·0747	3
4	797	42	·0527	604	31	·0513	373	22	·0590	180	16	·0889	4
5	776	47	·0606	602	34	·0565	381	25	·0656	178	21	·1180	5
6	743	42	·0565	587	32	·0545	390	18	·0462	165	13	·0788	6
7	716	40	·0559	565	32	·0566	397	29	·0730	162	13	·0803	7
8	690	55	·0797	539	44	·0816	385	34	·0883	151	17	·1126	8
9	643	63	·0980	509	48	·0943	358	36	·1006	140	21	·1500	9
80	601	62	·1032	480	48	·1000	344	37	·1076	128	19	·1484	80
1	563	60	·1066	450	48	·1067	331	39	·1178	109	11	·1009	1
2	511	46	·0900	408	38	·0931	306	28	·0915	96	13	·1354	2
3	456	67	·1469	378	58	·1534	288	40	·1389	85	15	·1765	3
4	385	51	·1325	315	41	·1302	249	26	·1044	70	11	·1571	4
5	334	46	·1377	281	42	·1495	223	34	·1525	65	16	·2462	5
6	300	34	·1133	254	28	·1102	203	18	·0887	55	11	·2000	6
7	266	45	·1692	234	38	·1624	187	29	·1551	43	10	·2326	7
8	212	39	·1840	189	36	·1904	153	28	·1830	34	13	·3824	8
9	172	34	·1977	153	32	·2092	127	30	·2362	20	4	·2000	9
90	134	42	·3134	113	36	·3186	91	30	·3297	14	3	·2143	90
1	92	27	·2935	79	22	·2785	65	19	·2923	15	5	·3333	1
2	65	18	·2769	57	15	·2632	50	11	·2200	10	3	·3000	2
3	46	18	·3913	41	16	·3902	37	14	·3784	7	4	·5714	3
4	28	6	·2143	24	5	·2083	22	5	·2273	3	1	·3333	4
5	24	11	·4583	21	10	·4762	17	7	·4118	2	1	·5000	5
6	14	2	·1429	11	1	·0909	10	1	·1000	1	1	1·0000	6
7	12	7	·5833	10	7	·7000	9	7	·7778	...	...	...	7
8	3	...	...	2	...	...	2	...	...	...	...	...	8
9	3	3	1·	2	2	1·	2	2	1·	...	...	...	9
100	1	1	1·	1	1	1·	1	1	1·	...	...	...	100
1	1	...	...	...	...	...	...	...	...	...	...	...	1
2	1	1	1·	1	1	1·	...	...	...	...	...	...	2
...	23,226	1,282	...	16,043	1,000	...	9,666	720	...	4,622	363	...	...

TABLE B.

Mortality of Life Tenants. Full Aggregate Tables. Graduated Values of q_x, μ_x and e_x.

Age	Females			Males			Age
	q_x	μ_x	e_x	q_x	μ_x	e_x	
40	·00615	·00610	32·90	...	...	...	40
1	·00631	·00626	32·10	...	...	...	1
2	·00649	·00642	31·31	...	...	...	2
3	·00669	·00661	30·51	...	...	...	3
4	·32959	·37976	1·68	·41816	·51483	1·21	4
5	·35790	·42071	1·50	·45056	·56922	1·07	5
6	·38790	·46612	1·34	·48440	·62955	·95	6
7	·41954	·51649	1·19	·51950	·69646	·85	7
8	·45273	·57236	1·04	·55565	·77068	·72	8
9	·48732	·63433	·91	·59256	·85299	·63	9
100	·52314	·70307	·77	·62994	·94431	·54	100
1	·55995	·77931	·62	·66739	1·04558	·46	1
2	·59746	·86389	·40	·70452	1·15794	·37	2

Females:

$$\mu_x = A + Bc^x$$

where $A = \cdot 00470471$

$B = \cdot 000022084$

$c = 1\cdot 109175$

and $\operatorname{colog}_{10} p_x = \alpha + \beta c^x$

where $\alpha = \cdot 00204323$

$\beta = \cdot 00001010554$

$c = 1\cdot 109175$

$\log_{10} c = \cdot 045000$

Males:

$$\mu_x = A + Bc^x$$

where $A = \cdot 01663511$

$B = \cdot 00002933567$

$c = 1\cdot 109175$

and $\operatorname{colog}_{10} p_x = \alpha + \beta c^x$

where $\alpha = \cdot 00722454$

$\beta = \cdot 00001342380$

$c = 1\cdot 109175$

$\log_{10} c = \cdot 045000$

TABLE B.

Mortality of Life Tenants. Full Aggregate Tables. Graduated Values of q_x, μ_x and e_x.

Age	Females			Males			Age
	q_x	μ_x	e_x	q_x	μ_x	e_x	
40	·00615	·00610	32·90	...	...	...	40
1	·00631	·00626	32·10	...	...	...	1
2	·00649	·00642	31·31	...	...	...	2
3	·00669	·00661	30·51	...	...	...	3
4	·00691	·00682	29·72	...	...	...	4
5	·00715	·00705	28·92	...	...	...	5
6	·00741	·00730	28·13	...	...	...	6
7	·00771	·00758	27·34	...	...	...	7
8	·00804	·00790	26·55	...	...	...	8
9	·00840	·00825	25·77	...	...	...	9
50	·00880	·00863	24·99	·02189	·02185	19·40	50
1	·00925	·00906	24·21	·02248	·02242	18·83	1
2	·00975	·00954	23·44	·02313	·02305	18·26	2
3	·01030	·01007	22·67	·02385	·02375	17·70	3
4	·01091	·01065	21·90	·02465	·02453	17·13	4
5	·01158	·01130	21·14	·02553	·02539	16·56	5
6	·01233	·01202	20·39	·02651	·02635	16·00	6
7	·01316	·01282	19·65	·02760	·02741	15·43	7
8	·01408	·01370	18·91	·02881	·02859	14·87	8
9	·01510	·01468	18·18	·03014	·02989	14·31	9
60	·01623	·01577	17·46	·03162	·03134	13·75	60
1	·01748	·01698	16·75	·03225	·03294	13·20	1
2	·01887	·01832	16·04	·03507	·03472	12·66	2
3	·02041	·01981	15·35	·03707	·03669	12·12	3
4	·02211	·02146	14·67	·03929	·03888	11·58	4
5	·02399	·02329	14·00	·04175	·04131	11·06	5
6	·02607	·02531	13·35	·04446	·04400	10·54	6
7	·02838	·02756	12·70	·04747	·04699	10·03	7
8	·03093	·03006	12·08	·05079	·05031	9·53	8
9	·03375	·03283	11·46	·05446	·05399	9·04	9
70	·03687	·03590	10·86	·05852	·05807	8·56	70
1	·04032	·03931	10·28	·06299	·06259	8·09	1
2	·04414	·04308	9·71	·06793	·06761	7·64	2
3	·04835	·04727	9·16	·07339	·07318	7·19	3
4	·05300	·05192	8·62	·07939	·07935	6·76	4
5	·05813	·05708	8·11	·08601	·08620	6·35	5
6	·06378	·06280	7·61	·09329	·09380	5·94	6
7	·07002	·06914	7·12	·10131	·10222	5·56	7
8	·07688	·07617	6·66	·11011	·11156	5·18	8
9	·08444	·08397	6·22	·11977	·12192	4·82	9
80	·09275	·09262	5·79	·13037	·13342	4·48	80
1	·10188	·10222	5·38	·14198	·14617	4·15	1
2	·11190	·11287	4·99	·15467	·16031	3·84	2
3	·12288	·12468	4·62	·16852	·17600	3·54	3
4	·13490	·13778	4·27	·18363	·19340	3·26	4
5	·14805	·15230	3·93	·20006	·21270	2·99	5
6	·16239	·16842	3·61	·21790	·23311	2·74	6
7	·17801	·18630	3·32	·23722	.25785	2·50	7
8	·19501	·20612	3·03	·25810	·28418	2·28	8
9	·21344	·22810	2·77	·28058	·31339	2·07	9
90	·23340	·25249	2·52	·30472	·34579	1·87	90
1	·25494	·27955	2·29	·33056	·38172	1·69	1
2	·27813	·30955	2·07	·35809	·42158	1·53	2
3	·30300	·34285	1·87	·38730	·46579	1·38	3
4	·32959	·37976	1·68	·41816	·51483	1·21	4
5	·35790	·42071	1·50	·45056	·56922	1·07	5
6	·38790	·46612	1·34	·48440	·62955	·95	6
7	·41954	·51649	1·19	·51950	·69646	·85	7
8	·45273	·57236	1·04	·55565	·77068	·72	8
9	·48732	·63433	·91	·59256	·85299	·63	9
100	·52314	·70307	·77	·62994	·94431	·54	100
1	·55995	·77931	·62	·66739	1·04558	·46	1
2	·59746	·86389	·40	·70452	1·15794	·37	2

Females:

$\mu_x = A + Bc^x$

where $A = \cdot 00470471$

$B = \cdot 000022084$

$c = 1 \cdot 109175$

and $\operatorname{colog}_{10} p_x = \alpha + \beta c^x$

where $\alpha = \cdot 00204323$

$\beta = \cdot 00001010554$

$c = 1 \cdot 109175$

$\log_{10} c = \cdot 045000$

Males:

$\mu_x = A + Bc^x$

where $A = \cdot 01663511$

$B = \cdot 00002933567$

$c = 1 \cdot 109175$

and $\operatorname{colog}_{10} p_x = \alpha + \beta c^x$

where $\alpha = \cdot 00722454$

$\beta = \cdot 00001312380$

$c = 1 \cdot 109175$

$\log_{10} c = \cdot 045000$

ABSTRACT OF THE DISCUSSION.

MR. C. CARPMAEL thought that the extraordinary vitality of female life tenants might make it necessary to revise the methods of calculating premiums for contingent policies if they were based as regards female counter-lives on the $O^{(af)}$ Table without modification. It was rather interesting to note the effect that the exclusion of option cases from the investigation would have. If these cases were excluded the ratios of actual to expected deaths for the first five years, for the second five years, and after ten years became 95·9, 93·6, and 102·5 ; showing a rather greater difference between the actual mortality in the first ten years and afterwards and reducing by almost one-half the difference referred to in the paper between the mortality in the first and second five years. With regard to the female experience he had found that a very rough graduation could be made by merely taking 85 per-cent of the expected deaths by the $O^{(af)}$ Ultimate Table. From that it followed by an application of Makeham's Law that the annuity values might be roughly found from that table by deducting two years from the age. That was a rough approximation and only applied to ages down to sixty, but it gave annuity values very close to the author's. The reasons why a Makeham graduation of the experience had been found possible might be merely that the experience was comparatively small and that only a small part of the whole range of life was included. With regard to the male experience he had found that in all age groups the actual deaths agreed very closely with the expected by the $O^{(am)}$ Ultimate Table— so closely, in fact, that the male life tenants might quite well be merely a random sample of the $O^{(am)}$ ultimate annuitants. If therefore more extended tables of joint life annuities were calculated, it would be perhaps safest to base them on the FemaleLife Tenants Table in conjunction with the $O^{(am)}$ Ultimate, because the rates of mortality by the latter table were almost throughout lighter than those given in the graduated table in the paper, and the annuity values were slightly higher. With reference to the author's ingenious method of approximating to annuity-values at other rates of interest than those for which tables had been calculated he thought that in the two cases given in the appendix most actuaries would be quite content with a first difference interpolation or extrapolation, as the case might be. The error would be only one and four respectively in the third decimal place, and it was impossible to rely on the experience to give that degree of accuracy. For substantially higher or lower rates of interest, the author's approximation might be useful, but it would be easy to find the values sufficiently accurately by means of the Carlisle Tables. For example, the value of an annuity at age 40 on the basis of the Female Life Tenants' Table at 10 per-cent could be found by taking the value by the Life Tenants' Table at 7 per-cent, finding the corresponding age by the Carlisle Table at 7 per-cent, and then using the Carlisle 10 per-cent Table ; the error would be less than 1 per-cent of the annuity value.

Mr. W. A. MOUAT JONES said that the ungraduated q_x for female life tenants showed that they might safely use the British Offices Ultimate Table with one or two years off the age, or preferably the British Offices Select Table. It was when one came to the graduated q_x that difficulties arose. Owing to paucity of data, the results at both ends of the table were not reliable. It was hardly conceivable that the q_{53} for female lives was approximately equal to the British Offices Ultimate q_{40}. Probably for ages 60 to 90 the rates of mortality of life tenants were fairly represented, but outside those ages the results were very doubtful. Before attempting to build up an elaborate superstructure of monetary tables, particularly for joint lives, it would be better to wait for the new annuitant experience which would probably be found to agree closely with the female life tenants' experience. The present annuity experience was thirty years old. He doubted whether an attempt to impose a rigid law like Makeham's on a small experience of this kind was justifiable. It would be very convenient if it could be done, but a Makeham graduation took no account of the weight of the data at various ages, and that was important when the data below 50 and over 90 were very small. He thought the Makeham graduation in the present case considerably distorted the rate of mortality at the two ends of the table. In the age group 85–89 the expected deaths (females) exceeded the actual by 25·57. The author spread it over the remaining age groups, with the result that he materially altered the rates for age 90 and over, giving a distinct twist to his curve. It would also be noticed that for ages 60–89—the most important part of the table from the point of view of purchasers of reversions—the positive deviations amounted to 45·82, whereas the negative deviations were only 18·45. It was worth noting that $\log_{10}c$ had been taken as ·045, the highest value for any known table. That seemed to account for the steepness of the curve at the higher ages. Would it not have been better to graduate with reference to some standard table, as suggested by Mr. Lidstone? He had graduated the experience roughly, using the $O^{(af)}$ Ultimate Table as a standard. He also had found that 85 per-cent of the female mortality expected by the British Offices Ultimate Table was roughly equivalent to the actual mortality shown by Mr. Lever's unadjusted data. The author's conclusion that there was practically no selection on the part of vendors seemed contrary not only to common sense but also to office experience. Usually a reversioner had an option of taking a loan or selling, and if he thought the life tenant was in bad health there was no question but that he would in the majority of cases take a loan. There were many cases—and they had increased during the war—where the reversioner or his representative was compelled to sell regardless of the state of health of the tenant for life, except perhaps when she was on her death bed. There were cases where the borrower had a loan and was unable to keep up his interest payments, sales by executors of deceased reversioners who wished to wind up an estate, sales by executors of deceased

borrowers who had no income out of which to pay the interest, and sales by trustees in bankruptcy. The reliability of the author's conclusions with regard to selection depended partly on the proportion of these cases to the normal sales. Again, many of the option cases were cases with powers of re-purchase only if the life tenant was living. These limited option cases had been brought under exposure as at the termination of the option period, although practically out-and-out sales. Consequently some of the data included in the full aggregate table, should be in the truncated aggregate after excluding the first 5 years; some included in the latter aggregate were of the class truncated aggregate excluding the first 10 years. This might have some effect on the results. In 74 female limited option cases which were on the books of a reversionary company during 1905 to 1919, the expected deaths during the period of option according to the $O^{(af)}$ Ultimate Table were 10, whereas the actual deaths were 9, so that their mortality might be considered as normal.

Mr. A. D. BESANT said that the actual deaths up to age 59 among the female life tenants exceeded the graduated deaths by no less than 20 per-cent. Up to age 60 the graduated death rates of females were less than one-half of those of males. Such a feature as that was absolutely unknown in any mortality table. It might be due to faulty graduation or to the paucity of the data, but he could not think it was a fact. In the Ultimate $O^{(am)}$ and $O^{(af)}$ Tables the mortality of females at age 50 was almost identical with that of males, at age 55 it was very nearly 90 per-cent and at age 60 about 75 per-cent. He would hesitate before giving up accepted tables which showed those relative mortalities and adopting tables which exhibited a female mortality of less than one-half that of male lives at the same age. Until publication of the contemplated new annuity experience he thought that the $O^{(am)}$ and $O^{(af)}$ Ultimate Tables afforded a very efficient instrument for the valuation of reversions, at any rate at the younger ages. At the older ages it was generally admitted that modern medical and surgical science had done much to increase vitality, and therefore that tables compiled twenty or thirty years ago did not accurately represent modern mortality at those ages. The difficulty could be got over by using the select tables at the higher ages. The author's data were compiled from the experience of seventeen life offices and four reversionary companies, and he (the speaker) was not sure whether the experience of life offices was quite the same as that of reversionary companies. In reversionary companies quite a large proportion of the transactions began as loans, and were converted into purchases when the borrower found that he could not pay his interest. Life tenants in such cases were not picked, selected lives. Impecuniosity was the governing factor by which the great majority of reversions came on the books, and he believed that was why the $O^{(af)}$ Ultimate Table did, as a matter of fact, produce fairly satisfactory results.

Sir JOSEPH BURN said that if the offices that had contributed to the experience which the author had dealt with were, with the

help of the author's tables, to examine their own results, he thought they would find that their purchases had not been anything like so favourable as they had hoped. They should look carefully into the matter, and see whether the business, even at present prices, was worth consideration in view of the many other very favourable investments which were open to any large insurance company. Funds invested in reversions were locked up for a long time, and there was no opportunity of changing the trust investments or selling out at a favourable time. In many cases they were investments which an insurance company would not hold on its own account.

MR. C. R. V. COUTTS mentioned that he had constructed a hypothetical reversionary business consisting of reversions on female lives distributed according to the exposed to risk in Mr. Lever's table, and had valued them by various methods. Mr. Lever's table gave a total value on a 6 per-cent basis of £121,800. The Carlisle Table with four years off the ages gave £123,600, a difference of $1\frac{1}{2}$ per-cent. The Carlisle Table was not such a bad approximation after all. The author had dealt at some length with the question of selection and had arrived at the extraordinary result that selection was a negative force, that a life tenant was a better life five years after the reversion had been bought than one of the same age at the time of purchase. There were only two possible explanations. One was that some of the reversions might have been bought on a special basis. There were instances of policies on damaged lives having been sold at a much greater value than the ordinary market value. The other possible explanation was that the data were too small to form any conclusion from.

MR. W. P. PHELPS said that he had employed the author's tables to test the valuation of 215 single life reversions, involving an amount equal to £550,000. Female lives predominated, being rather more than two to one. The Carlisle 6 per-cent value with 3 years off the ages of the female lives was rather over £321,000 and the value by the author's tables was nearly £317,000, a difference of under £5,000 or less than $1\frac{1}{2}$ per-cent of the amount involved. He wished it were possible to purchase reversions with the same degree of accuracy.

MR. E. W. PHILLIPS said that the tables given in the paper had been calculated from the experience of twenty-one offices, and it was useless to condemn the figures obtained merely because they were surprising. He could not agree with Mr. Mouat Jones's criticism of the graduation. As it had been found that selection was non-operative, he could see no reason why the convenient Makeham method of graduation should not be employed. It was suggested that they should wait for the new annuity experience; but if selection did not operate annuity tables would be unsuitable for measuring the mortality of life tenants. With regard to the practical use of the new tables, the want of a table of joint life annuities at equal ages could be got over by substituting one life for the joint lives and modifying the rate of interest. Whatever the rate of interest required when one life was substituted for two

or more lives, the author had provided in the Appendix of the paper a ready means of calculating functions at this rate by interpolation or extrapolation. For m joint lives the rate of interest had to be modified so that $v' = v \cdot s^{(m-1)}$. By a mere coincidence, the value of s was such that when one life was substituted for two, the addition of one half to the rate of interest per-cent gave the required modified rate to within one penny over a range extending from 5 per-cent to 7 per-cent, while for three lives the addition of 1 per-cent was within $2\frac{1}{2}d$. of the correct addition. The approximation involved in this simple and convenient rule would hardly affect an annuity-value given to three places of decimals, and would obviate any necessity for the construction of joint life tables.

Mr. F. L. COLLINS, in closing the discussion, said that in discussing the mortality of life tenants hitherto, actuaries had always been handicapped by want of sufficient data. Therefore they were the more indebted to Mr. Lever for his efforts to obtain an adequate body of facts. He thought it would be agreed that the 1,282 female deaths afforded a fairly reliable basis for most ages, but the 363 male deaths could hardly be numerous enough to give trustworthy results. In one respect Mr. Lever's figures were a model. They were certainly up-to-date. It should be remembered, however, that of the fourteen years covered by the experience, nearly a third represented the war period. The strain of the war, the influenza epidemic, increased taxation, must all have had some effect. It was wonderful that, in spite of those adverse influences, the female lives showed the highest standard of vitality hitherto published except at the extreme ages, where the data were too scanty to be reliable. The treatment of the data left room for some divergence of view. As had been pointed out by Mr. Besant, the graduated values of q_x showed some very remarkable differences between the male and female rates. That was due more to the abnormally low rates for the females than to the high rates for the males, as would be seen by comparing the Ultimate q's by the $O^{(af)}$ and $O^{(am)}$ or the corresponding rates shown by the Government (1910) Annuity Experience:

Adjusted Values of q_x.

Age	Mr. Lever's Life Tenants		British Offices Annuitants Ultimate		Government 1910 Annuitants	
	Female	Male	$O^{(af)}$	$O^{(am)}$	Female	Male
50	·00880	·02189	·01533	·01544	·01289	·01643
60	·01623	·03162	·02179	·02853	·02120	·02944
70	·03687	·05852	·04358	·05926	·04374	·05994
80	·02975	·13037	·10793	·12904	·10318	·12926
90	·23340	·30472	·24690	·27603	·24847	·27543

These results led one to examine more closely the lower ages in the case of the females, where, of course, the data were very scanty.

The comparison of the actual deaths with the expected was as follows :

Ages	Actual Deaths	Expected Deaths	
		Mr. Lever	$O^{(af)}$ Ultimate
40–44	2	1·25	2·14
45–49	8	3·72	6·53
40–49	10	4·97	8·67

showing that the actual deaths were more than double what Mr. Lever's graduated table would lead them to expect in that age group. The exposed to risk of the office annuitants (females) were about eight times the exposed to risk of the life tenants (females).

It seemed probable that a graduation of the ratios $\log \frac{p_x}{p'_x}$, as suggested by Mr. Lidstone, *J.I.A.*, vol. xxx, p. 212, by reference to the $O^{(af)}$ Ultimate Table, would give very different values at the lower ages and respond to the usual tests for graduation.

With regard to the conclusions at which Mr. Lever had arrived, the absence of selection on the part of the vendor was almost incredible. He thought, with other speakers, that the adage, "A bad loan often becomes a good purchase", was the key. Where the life was bad, the reversioner would not give up the idea of raising money altogether. He would borrow in the hope of being able to pay off later. Hope told a flattering tale. Impecuniosity eventually forced a sale. To regard that type of case as being homogeneous with an out-and-out sale seemed to him to be wrong. The same argument also applied to options to repurchase, as Mr. Jones had pointed out. This was the answer to Mr. Lever's conclusion that annuity tables were not suitable for valuing reversions because annuitants exercised selection against the office and vendors of reversions did not. It was true that the selection exercised by the annuitants was far more potent than that which the vendor of a reversion could exercise. An annuitant could, if necessary, be medically examined for his own satisfaction, whereas a reversioner might find it difficult to obtain definite information about the health of the life tenant. But if the life tenant were known to be dying, obviously the reversioner would not sell. The selection exercised was only a question of degree. Mr. Lever's other reason for not using annuity experience in valuing reversions was an additional argument for pushing on the new annuity experience of British Offices.

He did not think that the Carlisle mortality had actually been used very much of late years. The monetary tables, for which they were indebted to the industry of the late Mr. David Jones, merely afforded a useful practical method of obtaining approximations,

High rates of interest now in use were not tabulated elsewhere. But actuaries had not really regarded the Carlisle Table as a mortality basis. Even the deduction of three years for females had long been considered out of date.

The PRESIDENT thought that whatever might be the views of the members as to the merits of the arguments put forward by Mr. Lever, they would be unanimous in agreeing that they owed him a very hearty vote of thanks. The paper was eminently workmanlike and left nothing to surmise, though there might be something more to be said on the question of the most appropriate methods of graduation in connection with such an experience. With reference to the Carlisle Table almost every solicitor, accountant and estate agent had some table based on the Carlisle and used very often to confute the actuary who had given a really scientific estimate of the value of some risk. If the discussion on Mr. Lever's paper merely established the fact that different Mortality Tables were suitable and indeed were imperatively necessary in different circumstances, that in itself would be something of value in educating the public on a point on which at present there was a great deal of lamentable ignorance. A few days ago an eminent body of mathematical workers, not of this country, obtained Part II of the Institute Text-books, and in connection with some calculations for a superannuation fund for highly selected lives in a carefully selected service they took the H^M Table at the end of the book and, seeing it was headed "Life Table", presumed it was the life table and used it. That was the kind of ignorance he wished to dispel.

MR. LEVER, in reply, said that the criticisms of his paper might be grouped under four main headings—(*a*) that certain of the results were rather surprising in their nature, (*b*) that the results did not in all cases bear out the experience of individual members, (*c*) that the graduation did not attain perfection in all respects, (*d*) that it would have been better to have waited before calculating monetary functions. With regard to the surprising nature of some of the results, if the investigation showed that certain preconceived ideas were not well founded its value was enhanced. As regards the second group of criticisms, a complete experience could not be expected to conform exactly to others which from their nature were only partial. The graduation he had always expected to be criticised, but in spite of the points urged against it, the force of which he fully recognised, he was prepared to defend it, especially as the practical effect of any modification would be negligible. So far as the male table was concerned, the deaths were few, and he would be the last to suggest that the rates were more than a broad indication of the mortality.

With regard to selection, he had not stated that there was none but that the data did not show any. In all the criticism on this point no one had been prepared to say that the data, as such, showed that selection existed to a marked degree. His main object was to show that so far as those particular data were concerned,

the labour of constructing select tables would be entirely misspent and that the mortality could be accurately represented by an aggregate table. He was inclined to agree with Mr. Collins that the war might have had a certain effect in the case of female lives, but he hardly thought the effect would have been sufficient to make a material difference to the results. The effect, if any, would be in the way of masking the light mortality that had been brought out.

On the Relation between the Course of Wholesale Prices of Commodities and the Market Value of Various Classes of Securities. By S. J. PERRY, F.I.A.

[Submitted to the Institute, 28 November 1921.]

THE history of life insurance presents some curious changes in the character of the factors which have from time to time taken precedence as matters of anxiety to those responsible for the successful conduct of the business. In the early days it was anticipated that it was the rate of mortality which would give the greatest possibility of variation, and which was most likely to undermine the foundation of the structure. In time, however, this bogey was laid by the heels. Then, in the final years of last century the fall in the rate of interest was a matter of great concern in view of the persistent fall which had occurred over a long period, the danger being that the rate earned on accumulated funds would fall below that necessary to maintain solvency.

Since that time, however, a complete change of front has taken place. There is now no prospect of the rate of interest obtainable falling to a dangerously low level; but by reason of the constantly increasing yield obtainable from first-class securities, the problem is how to avoid loss by depreciation in the market value of securities in which funds are invested.

It must be admitted that the successful or unsuccessful management of the funds of a life insurance institution gives a very large scope for profit or loss. Profit arises mainly from three sources: first, excess loading over expenses of management; secondly, more favourable mortality than that assumed in the valuation; and thirdly, through a higher rate of interest being earned than that which is assumed. Whilst anxiety may have been felt in respect of all these, in point of

fact it is undoubtedly heavy depreciation which has temporarily eaten into the bonus-earning power of many offices, and though I feel there are many far better qualified than myself to deal with the subject, I am prompted to offer this short paper for the consideration of the Institute in the hope that criticism and a discussion of the subject will bring forward information and suggestions of practical use to the profession.

It is a commonplace to state that the return obtainable for the use of capital depends largely on the ordinary law of supply and demand. When the demand for capital is greater than the supply the yield will rise, and during periods of warfare and of great development such is the case.

The trade cycle of increasing and decreasing trade is a well-known phenomenon of our economic organization. Trade increases and commodity prices rise. Profits expand, and the good use that can be made of capital induces an increased demand for it; therefore the rate of interest obtainable rises. Then over-speculation precipitates a crisis of some description, commodity prices fall, the use of capital becomes less remunerative, and the yield falls again.

The effect of this cycle on fixed interest securities is to cause a fall in the market value of existing stocks as the yield obtainable from new capital rises. The shares of trading concerns making good profits during this time have two forces operating on them from opposite directions: (1) the increasing profits tending to produce a rise in the market value, and (2) the increasing yield from new capital tending to produce a fall in the manner outlined for fixed-interest stocks.

Let us look at the matter from a slightly different point of view, the use of capital for the purpose of trade and commerce, and the competition for the favour of the investor which exists between dividend paying stocks and shares and fixed interest bearing securities. Merchants, manufacturers and trading concerns normally hold stocks of commodities to enable them to carry on their businesses. When prices rise it follows that additional profits are made on the stocks of commodities in hand, and so far as trading companies are concerned the result shows itself in due course, when the accounts are published, in increased dividends and profits. Increased dividends on ordinary shares, however, do not mean an increased return on the fixed-interest securities. A desire on the part of the investor to share in the prosperity of trade results in an

endeavour to sell interest-bearing stocks and to re-invest in directions that promise better returns. So there arises a competition between the two classes tending to increase the market value of the share and to decrease the value of the interest-bearing bond.

Now when prices of commodities begin to fall the stocks of commodities held by traders have to be disposed of at a smaller margin of gain or even perhaps at a loss; profits fall and with lean prospects the effect is a drop in the market values of trading shares. If a crisis appears, commitments have to be met and the securities with a solid contract behind them—that is, fixed-interest securities—are sold to meet the commitments, causing them also to fall in price, so that all classes of securities tend to fall in value together. However, when the situation has been met, attention is once again directed to those securities which hold the better outlook and, trading prospects being unfavourable, the investor turns to the fixed-interest stock which begins to recover.

This briefly outlines the course of investment values during a cycle of trade.

There is one other consideration I should like to state at this stage with reference to the return on capital. This may be looked on as divisible into two parts.

(1) Payment for the use of the money.

(2) Payment for the risk of failure on the part of the borrower to fulfil the contract, or for risk of loss of the capital.

In practice this is translated into the fact that the more speculative a security is, the higher is the yield anticipated from it. It is obvious that in first-class fixed-interest stocks, such as national funds and corporation stocks, the allocation to the second of these divisions is practically negligible, whilst in securities lower down the list it is considerable. It is this estimate of the balance between risk of loss and probable return on the capital invested, so as to obtain the largest possible return for the loan of it, which fixes the market price of a security. As the market view of the credit of a borrower or concern becomes more favourable, so the yield falls, that is, the price of the security increases.

It is a sound principle of insurance investment practice as far as possible to avoid the second element, the aim being to invest the funds without risk. Unfortunately this does not

avoid depreciation. The only certain method of avoiding serious depreciation would seem to be that of investing in redeemable stocks with short terms to run to the maturity date. When a purchase is made of £100 nominal of say 3 per-cent irredeemable stock, what is purchased is not the right to £100 at any time, but a perpetual annuity of £3, and it is the annuity actually for which the market price is quoted. Even though the contract be fulfilled to the letter by the borrower it does not avoid the variation in value of the perpetual annuity, which variation is in inverse proportion to the long term estimate of the yield to be obtained for the loan of capital.

If therefore a sound opinion can be formed of the probable course of the yield to be obtained for long term investment it is of the utmost value, enabling the weight of funds to be thrown into long term or short term securities as the case may be; for should the yield fall long term investments will rise in value, and should the yield rise short term investments will at least avoid depreciation.

When considering the probable course of variation in the yield obtainable on investments one's mind naturally turns to the Bank Rate, but though a change in the Bank Rate often inaugurates a movement in the prices of Stock Exchange securities, it is essentially a guide only to the short term value of money, and further it does not give sufficient notice of any probable change for full advantage to be taken of it.

The leading daily papers, too, often point out in their financial columns the causes of minor fluctuations but seldom is a long view taken of the situation.

For the purpose of tracing the variations in the yield obtainable for long term investment Consols appears to be an ideal stock. Price records are available over a very long period and are tabulated in various places, whilst to all intents and purposes it is and has been irredeemable for the whole term of its existence except for a short period from 1880 to 1888. The yield represents as nearly as can be obtained that for the use of the capital only, as the risk of failure of Government to fulfil its contract is negligible.

The yield plotted in Diagram I has been calculated from the average price of each year up to 1888, whilst the interest was 3 per-cent, by the formula

$$\frac{3}{\text{Average Price}} \times 100$$

For the years 1889 to 1903, to allow for the period during which the interest was $2\frac{3}{4}$ per-cent, a deduction was made from the tabulated average price of an annuity-due of 5*s*. for the unexpired period at $2\frac{1}{2}$ per-cent, the adjusted price thus obtained being converted by the formula, corresponding to that given above, namely :

$$\frac{2\cdot5}{\text{Average Price}} \times 100$$

which also has been used for the years 1904 onwards. As the yield from 1889 to 1903 varied by such small limits from $2\frac{1}{2}$ per-cent the error introduced is negligible for the purpose of this paper, especially as it is direction rather than absolute variation that is in question.

For the purpose of tabulating the level of commodity prices (see also Diagram I), I have chosen the *Statist* Index Number in continuation of Mr. Sauerbeck's figures. This is based on the wholesale prices of 45 commodities and seems the best for the purpose required. Money being merely the machinery of exchange, prices of commodities in general tend to rise or fall together. Wholesale prices would appear to have an advantage over retail prices as any variation in the general level must affect wholesale prices rather earlier than it would retail prices. The *Statist* numbers are available back to 1846. For the years previous to this the figures of the late Professor W. S. Jevons,* based on 40 commodities, have been used, adjusted proportionately to equate the levels of the two numbers for 1846.

Passing to a consideration of Diagram I, it will be seen at once that the curve representing the yield of Consols, and that representing the course of commodity prices rise and fall together, a lead of one or two years usually being given by commodity prices. The periods during which sympathetic movement did not occur are few, and I would draw attention to the comparison of the following periods :

* Investigations in Currency and Finance, p. 136.

Periods of Downward Movement			Periods of Upward Movement		
Commodities	Yield of Consols	Years lead given by Commodities	Commodities	Yield of Consols	Years lead given by Commodities
1809–13	1812–14	3			
1814–16	1815–18	1			
			1816–18	1818–20	2
1818–24	1820–24	2			
			1824–25	1824–26	...
1825–29	1826–29	1			
			1829–31	1829–31	...
1831–33	1831–35	...			
			1833-36	1835–36	2
1836–38	1836–38	...			
			1838–39	1838–41	...
1839–44	1841–44	2			
			1844–47	1844–48	...
1847–49	1848–50	1			
			1851–54	1852–55	1
1854-58	1855–58	1			
			1858–64	1858–66	...
1864–68	1866–68	2			
1877–87	1878–89	1			
			1887–90	1889–91	2
1890–96	1891–97	1			
			1896–1900	1897–1901	1
			1908–20	1908–20	...
1920-21	1920-21	...			

I had at first intended to make a short study of the movements in the values of securities in post-war periods, but in face of the general sympathetic movements between the two curves it seemed by far the better plan to deal with the whole period.

It will be noted that the commodity curve is more sensitive than that for the yield of Consols, which is an advantage if practical use is to be made of the relation between them.

Bearing in mind that a fall in the yield means a rise in the price of Consols it will be noticed that in agreement with the earlier statement that a fall in commodity prices may cause the value of fixed interest-bearing securities also temporarily to fall, the average lead given by commodity prices when they have fallen is $1\frac{2}{13}$ years in the thirteen downward movements set out above, whilst it is only $\frac{8}{11}$ths of a year in the eleven upward movements. In other words, the lag for a period of recovery in security values is generally longer than that for a period of fall.

The only appreciable periods during which sympathetic movements did not take place are :

(1) 1807–09 for commodities	compared	with	1807–12	for Consols.
(2) 1868–77	,,	,,	1868–78	,,
(3) 1900–08	,,	,,	1901–08	,,

The first of these was during the Napoleonic Wars and whilst war wastage may have caused commodities to rise, optimism due to war successes may have caused the national credit to improve in much the same way that short-lived optimism brought about a rise in security values during 1918. In the second period the marked variation was in the years 1871 to 1875 when a rise and fall in commodity prices did not influence the yield of Consols. A flow of money into this country from the Continent for safe investment on account of uneasiness induced by the Franco-German War would have caused just such a disturbance of the normal law. From 1901 to 1908 Consols fluctuated but increased their yield over a period when commodity prices fell and rose again to 1907, and then sustained a sharp fall. The curves cannot during this period be said to act in unison. It will be remembered that the list of investments available to trustees was very largely increased in 1888–89. Trustees are notoriously apathetic whilst matters go well, and it may have been only when capital loss became evident that full advantage was taken of the extension of the available list of investments to broaden the basis of trust funds investments in the hope that further capital depreciation would be avoided. Otherwise I would suggest that the great fall in yield which reached its lowest point in 1896 induced a search for higher yielding investments, and that the development of our Colonies and realization of the inter-dependence of the interests of all nations introduced a new element of competition with the leading home securities, causing a rise in the yield obtainable from them. However, even during these years there would appear to have been causes affecting the yield in both directions, causing the fluctuations which are plotted.

From 1881 to 1888, too, Consols do not follow the commodity curve, but this is explained by the fact that the price was approximately par, and as the stock was then redeemable at six months' notice it could not rise appreciably above that figure. As soon as the conversion of 1888 had taken place the yield dropped to the point it would have reached had it been free to follow the commodity curve, and I have therefore had no hesitation in assuming that the true yield curve would have followed a normal course.

The principal panics, crises and wars are indicated at the top of the diagram.

In order to present a more detailed view of the later portion of the curves, in Diagram II I have set them out in quarterly averages from 1910 onwards. The break in the curve representing the yield of Consols is for the period from July 1914, until the end of the year, when the Stock Exchange was closed. The minor fluctuations are greater as would be expected. There is a marked variation in the direction of the two curves from the second quarter of 1917 to the last quarter of 1918 which would appear to be due to waves of optimism caused in the first place by the entry of the United States of America into the War, which produced a fall in the yield of Consols. This, however, showed a tendency to expend its energy towards the end of 1917. Then in 1918 a further rise in the value of Consols occurred, culminating in the price reached soon after the Armistice was signed. Otherwise the general trend is the same and whilst in Diagram I. the final change of direction of both appears to have taken place at the same time, the commodity curve is here seen to have a lead of approximately six months.

There would, from the foregoing, appear to be a sound case for periodical revision of investment policy in accordance with any decided change that may occur in the course of commodity price levels. However, it must be admitted that any attempt exactly to forecast the future is fraught with danger. The past results which have been recorded deal with a period during which little systematic advantage has been taken to exploit the situation.

Whilst funds are increasing it is not wholly disadvantageous for the Offices to have an increasing yield. Prospects of future profit become more favourable when current depreciation has been met, but a violent attempt to put enormous quantities of long term securities suddenly on the market might bring about depreciation of capital value too heavy to be borne. On the other hand systematic buying of large amounts of long term securities would rapidly put up the price and a decreased yield would result, though where large amounts of long term securities are already held there would be compensation in the appreciation which would accrue in respect of current holdings.

A solution of the problem therefore would appear to lie in the direction of a balance between loans on mortgage, investments with the absolute sheet anchor of a fixed date of redemption a

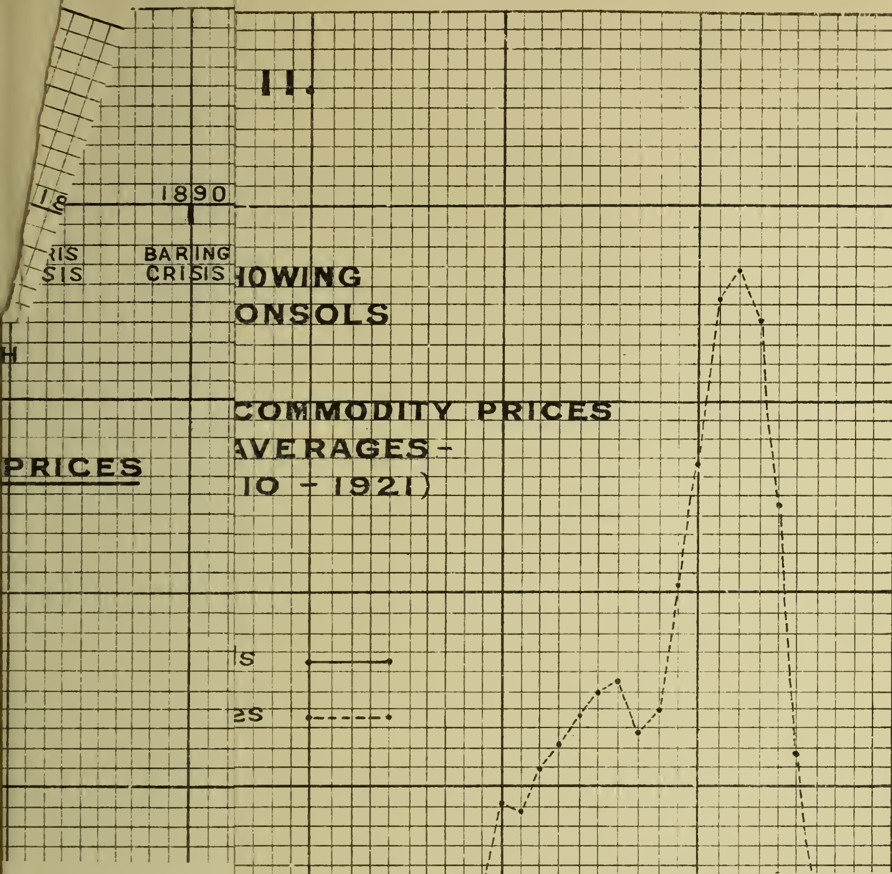
few years ahead only, redeemable securities with moderately long periods to run and a correspondingly moderate chance of appreciation or depreciation, and irredeemable securities with the greatest chance of appreciation or depreciation, throwing the balance by a revision of investments from time to time in the direction calculated either to bring a profit from appreciation or to guard against depreciation.

In tracing the course of the changes in the yield of Consols I have dealt only with a security where the guarantee for fulfilment of the contract is beyond question. That is to say, I have as far as possible omitted consideration of the second portion into which the return on capital may be divided, referred to on page 44, namely, payment for the risk of failure to fulfil the contract or for the risk of loss of capital.

These conditions can only be said to apply to home stocks of the highest descriptions, and then only approximately, where the credit concerned is that of the nation or of a substantial community, whose commitments are comparatively small. In other stocks and debentures the question of credit will always arise, and is at times sufficiently powerful as a second force to overcome the variation in price caused by a change in the yield demanded by the market.

Further, in the case of securities based on foreign currencies or on gold there will be disturbances due to the course of the exchanges.

In Diagram III I have traced the price movements of various classes of securities from 1887 to the present time. They are taken from the figures relating to Stock Exchange Values published in the *Bankers' Magazine* and for the purpose of easy comparison I have reduced them all to a common basis by taking the 1913 value as 100 and working backwards and forwards from that date. The series was commenced regularly in 1887, revisions of the lists of securities being made in 1895 and 1906. For December 1895 and 1906 results were given for both the then old and new bases, so that it was possible to pass from one to the other without disturbance of the index numbers. It is merely as if rearrangements of investments were made at these dates. To have attempted to take representative securities in any other way and trace them back would have introduced a possible fallacy as it would have been humanly impossible to have remained uninfluenced by the present day position and the intervening changes in the value of a security. With the


II.
1890
BARING CRISIS
HOWING
ONSOLS
COMMODITY PRICES
AVERAGES -
IO - 1921)
PRICES

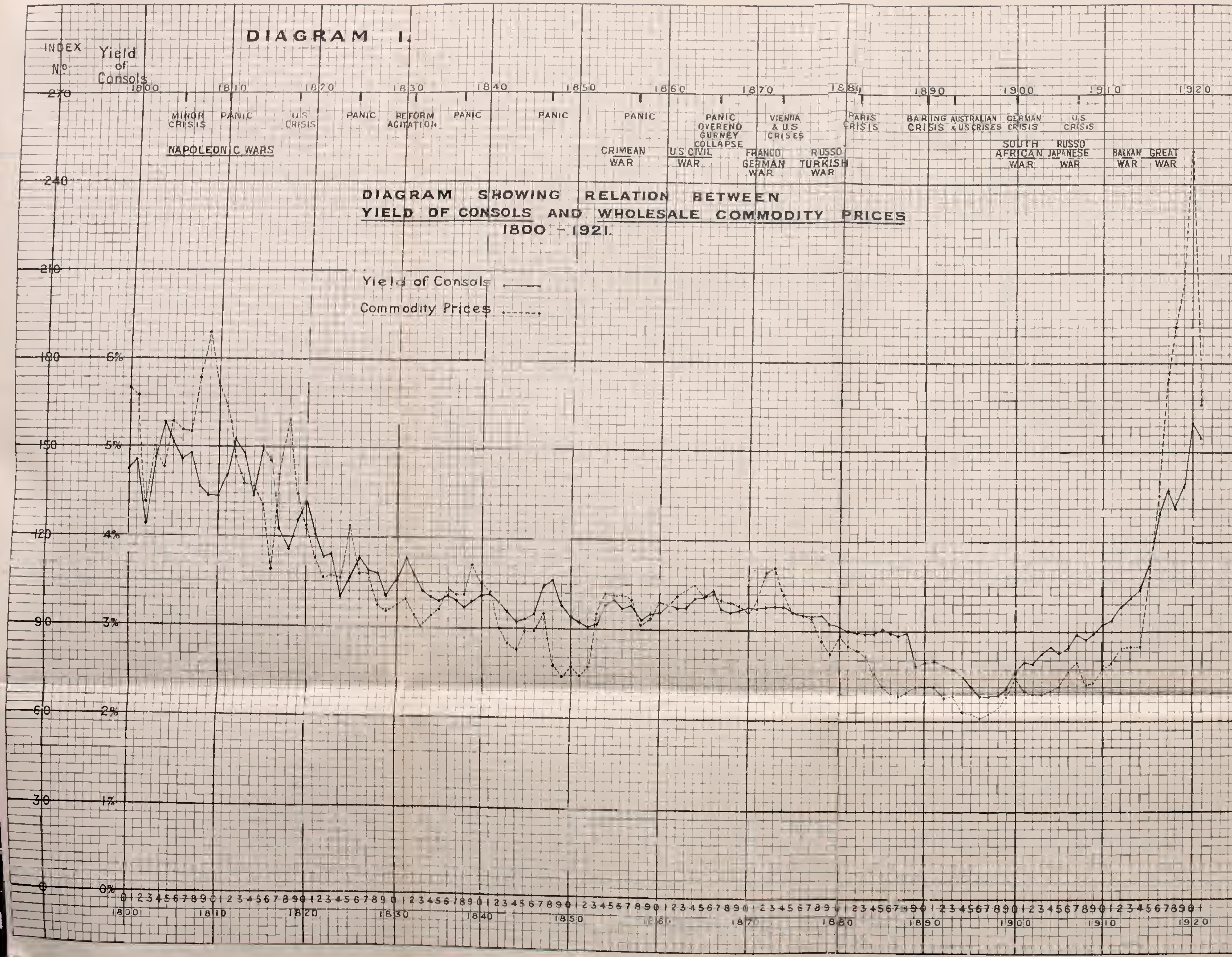


DIAGRAM I.
DIAGRAM SHOWING RELATION BETWEEN YIELD OF CONSOLS AND WHOLESALE COMMODITY PRICES
1800 - 1921.
INDEX Nº
Yield of Consols
Yield of Consols
Commodity Prices
MINOR CRISIS
PANIC
U.S. CRISIS
PANIC
REFORM AGITATION
PANIC
PANIC
PANIC
PANIC OVEREND GURNEY COLLAPSE
VIENNA & U.S. CRISES
PARIS CRISIS
BARING CRISIS
AUSTRALIAN & U.S. CRISES
GERMAN CRISIS
U.S. CRISIS
NAPOLEONIC WARS
CRIMEAN WAR
U.S. CIVIL WAR
FRANCO GERMAN WAR
RUSSO TURKISH WAR
SOUTH AFRICAN WAR
RUSSO JAPANESE WAR
BALKAN WAR
GREAT WAR
270
240
210
180
150
120
90
60
30
0
6%
5%
4%
3%
2%
1%
0%
1800
1810
1820
1830
1840
1850
1860
1870
1880
1890
1900
1910
1920

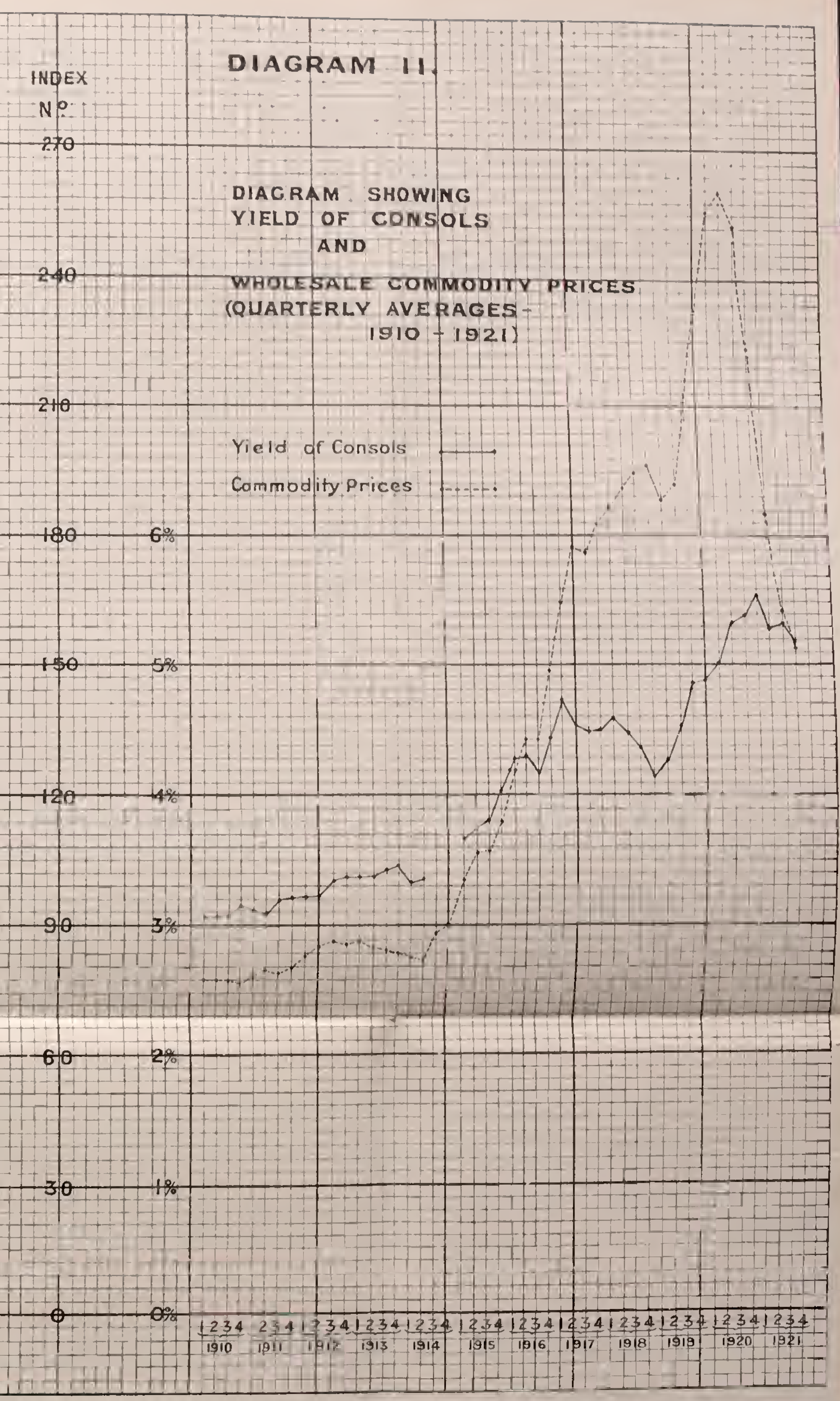
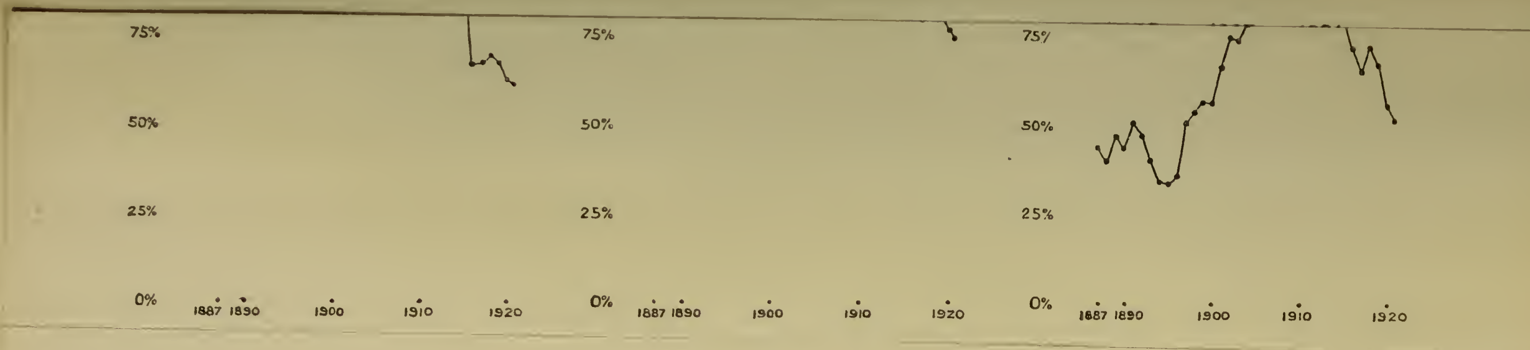


DIAGRAM II.
DIAGRAM SHOWING YIELD OF CONSOLS AND WHOLESALE COMMODITY PRICES (QUARTERLY AVERAGES - 1910 - 1921)
INDEX Nº
Yield of Consols
Commodity Prices
270
240
210
180
150
120
90
60
30
0
6%
5%
4%
3%
2%
1%
0%
1910
1911
1912
1913
1914
1915
1916
1917
1918
1919
1920
1921

DIAGRAM III.



75%
50%
25%
0%
1887 1890
1900
1910
1920
75%
50%
25%
0%
1887 1890
1900
1910
1920
75%
50%
25%
0%
1887 1890
1900
1910
1920

DIAGRAM III

DIAGRAM SHEWING THE COURSE OF PRICE MOVEMENTS OF VARIOUS CLASSES OF SECURITIES - 1887 TO 1921
1913 VALUE = 100%

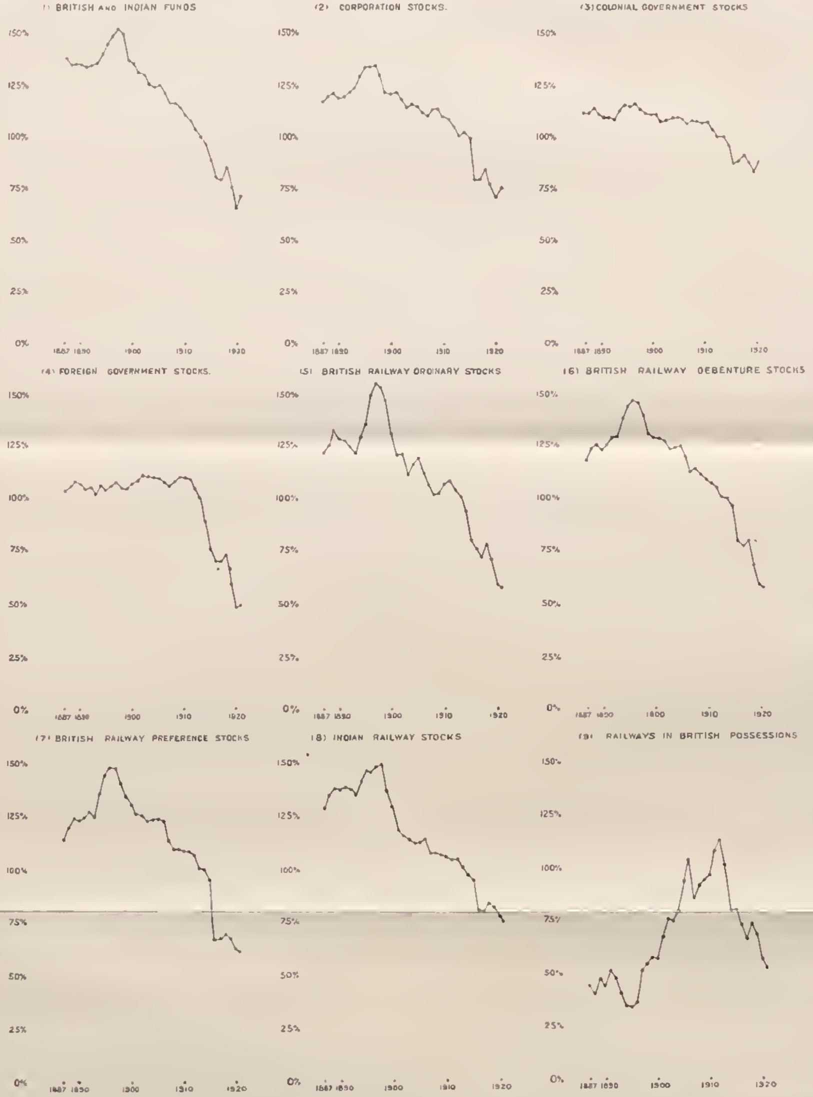


DIAGRAM III (Contd)

exception of some minor alterations made in the 1887–95 period there appears to have been no change in the lists of securities dealt with, so that it may be assumed that we have a representation of what would have taken place had capital been invested in the various classes by selection, as has to be done in practice. The figures represent the actual ones for December in every year, not averages as in the case of Diagram I, and this should be borne in mind whenever comparisons are made with the commodity index numbers. The only case where the figures are not those for December are the points plotted under 1914, which actually are those for January 1915 when the Stock Exchange was re-opened. For the purpose of comparison I have re-plotted the commodity index number from 1885 but have reversed it.

I have included (1) British and Indian Funds, (2) Corporation Stocks, (3) Colonial Government Stocks and (6) British Railway Debenture Stocks to illustrate the sympathetic movement due to change of yield in all first-class stocks. The three figures for British Railway Stocks illustrate to some degree the working of the element of credit. Commercial Industrial Shares, while not in themselves suitable as investments for insurance funds, show clearly some tendencies to follow variations in commodity prices, which tendencies in a lesser degree would be passed on to debenture issues of the trading companies. The iron, coal and steel shares show special features.

With the exception of commercial industrial shares the rise in values from 1917 to 1918 not being justified, but based only on sentiment, could not be sustained.

More particularly the following comments suggest themselves with regard to the various figures :

(1). *British and Indian Funds.*—These naturally show the same characteristics as Consols in Diagram I. As previously mentioned the immense widening of the list of securities available to trustees may have caused the failure to follow the commodity prices during the period 1900 to 1908.

(2). *Corporation Stocks.*—Exactly the same tendencies are shown as for British and Indian Funds, but the movements are not nearly so large. Such first class securities as these would naturally be subject to the same influences as national funds, but it seems probable that the relatively smaller movements are due to the existence of redemption dates, which would have a steadying effect both in upward and downward directions.

(3). *Colonial Government Stocks.*—Here again the same characteristics exist but the movements are yet relatively smaller. Fixed redemption dates would account for the main part of the comparative steadiness, though the rising credit of the various colonies will probably have played a part during a period of falling values. The commodities prices minima and maxima of 1887, 1891, 1896, 1900, 1903, 1907 and 1920 are followed by stock exchange maxima and minima of 1889, 1893, 1897, 1902, 1905, 1907 and 1920 respectively.

(4). *Foreign Government Stocks.*—These obviously are subject to many influences unconnected with British stocks and shares, and generally speaking the London prices would conform to those of the foreign bourses at the current rates of exchange. The curve is totally different in character to those preceding, but except for the effect of depreciated exchanges being shown in the extremely low level reached in 1920, the components would be too heterogeneous for useful conclusions to be drawn.

(5). *British Railway Ordinary Stocks.*—These appear, in the period under review, to follow more closely than first-class fixed-interest stocks the commodity curve. The upheaval of the war period produced its effect and under the guarantee arrangement between Government and the Railway Authorities the ordinary shares became more akin to fixed-interest stocks. Being subject to Parliamentary control the railways were prevented from raising their rates to a level corresponding to the general rise in commodity prices and with other trading monopolies subject to State control they suffered. These stocks would appear to have largely the disadvantages of the fixed-interest stock without the advantages of the normal trading share.

(6). *British Railway Debenture Stocks.*—These have followed a course between those of national funds and railway ordinary stocks. One can trace two forces at work, the increased or decreased yield general to all securities and the rise or fall in the credit of the railways, as reflected in the movement of the ordinary stocks.

(7). *British Railway Preference Stocks.*—The credit factor is naturally stronger in this case than with the debenture stocks and the course of values approximates nearer to that of the ordinary stocks.

(8). *Indian Railways.*—These have followed a very similar course to home railway ordinary stocks, though within smaller limits, which seems curious when comparison is made with

"Railways in British Possessions" which follow. Large developments must have taken place in India. No doubt stores and materials would mainly be shipped from this country and the traditions and principles of Indian management would be very similar to those of the home railways, and the stock would mainly be held in this country, but I am not able to judge whether these factors are sufficient to produce similar tendencies in price movements.

(9). *Railways in British Possessions.*—Here there is a totally different curve, probably due to the immense developments which took place in the Colonies during the period before the War, resulting in larger net earnings and rising stock values and credit of the railways concerned. Since 1914 the values of the stocks have followed the same course as those of other securities by reason of the great rise in the yields obtainable from loanable capital. Railway difficulties too appear to be world wide and falling credit has prevented recovery since 1920.

(10). *Commercial Industrial Shares.*—I have mentioned earlier the effect of the trade cycle in trading concerns, pointing out that rising prices mean increased profits with consequently higher values for the shares, until a crisis lowers prices of commodities, making commercial difficulties, with a consequent drop in the values of all securities. Comparing prices of commodities with commercial industrial share values, it will be noticed that rising prices of commodities to 1900 corresponded with rising values of the shares. Falling prices brought a sharp reaction in 1901 with rises again to 1907, when there was again a reaction. Then the movement again reversed, until about 1910 the rising yield of gilt-edged securities brought about another reaction which continued until war broke out. Then the uncertainty reduced the market values of the securities until 1915 when the tremendous rise in commodity prices brought tremendous profits to traders, overbearing the general rise in yield and causing rises in the share values until 1919. Then in the second half of 1920 the breaking of the crisis and consequent fall in commodity prices brought heavy decreases in share values, which the fall in market prices discounted, allowing for a small recovery since.

(11). *Iron, Coal and Steel Shares.*—This curve presents special features, being affected very much apparently by demands for war purposes, the three peaks of the figure corresponding in dates with the South African War, the Russo-Japanese War,

and the Great War respectively. A cursory examination of the curve for wholesale prices of minerals indicates that it is of much the same form as this. There does not, however, appear to be any lag in the share value curve, but if anything share values anticipate the movement of commodity prices.

I have endeavoured to give examples of the movements in the market values of the main classes of securities and it will be seen that in all except those of the highest class other influences than commodity prices have great sway. I have not included any class with special reference to its stability for purpose of the investment of insurance funds, but causes which affect the values of ordinary shares invariably affect the credit behind, and therefore the values of prior charges; and points which can be brought out with reference to any class of security are accordingly of use.

I have purposely avoided entering into currency questions and the causes lying behind variations in the prices of commodities, preferring to concentrate on the clear practical relation which appears to exist between the course of commodity prices and the market value of the higher classes of securities.

Further, I have refrained from dealing with the merits of individual securities with the exception of Consols, which is taken as representative of the highest class. It is general principle rather than detail which should I think be brought before the Institute.

Finally I would like to suggest that, though I have dealt with what is probably the most obvious example of its kind, with adequate research other factors may be discovered which would be of practical use with regard to the formation of investment policy, especially with reference to classes of securities where the higher yields are obtainable.

ABSTRACT OF THE DISCUSSION.

MR. A. J. C. EDWARDS said that in his opening remarks the author exhibited symptoms of the disease for which he prescribed remedies, namely, the implicit belief in the permanence of transitory conditions. It was true that recently the problem had been how to avoid loss by depreciation of investments, but he would be inclined to modify the author's preceding statement and say only that there was no *immediate* prospect of the rate of interest falling to a low level. In life insurance it was necessary to take a long view, and interest rates might have actually topped the crest of

the wave. Nor did he agree that the rate of mortality was finally disposed of. The fall had been assumed to be permanent, but the next investigation might show a different result. The war and the influenza epidemic must have affected the vitality of the nation, and with Russia, Austria and Ireland in their present respective conditions, mortality might yet prove a very disturbing factor. It was possible also that the author's statement of the features of a trade cycle might require to be revised in the light of Sir William Beveridge's new theory of the periodicity of harvests.

It was not clear to him how the movements of commodity prices were to be applied in the formation of investment policy. The purchase of long-term investments when commodity prices were falling would be the normal procedure, since such securities would then be appreciating in value. Similarly, when commodity prices were rising the normal procedure would be to buy short-term investments, to avoid the risk of depreciation. The difficulty would lie in determining whether the change in the direction of the curve of commodity prices was merely temporary or not, and it would need to be a very persistent change of direction to warrant a systematic revision of investments. Rigidly followed, the assumed investment policy would ultimately lead to a continual piling up of long-term investments. It should be borne in mind that in normal times the supply of suitable short-dated redeemable securities was liable to fall short of the demand.

The author compared Indian railway stocks with British railway ordinary stocks as contrasted with stocks of railways in other British possessions. The truer comparison would be with a combination of British and Indian funds and British railway ordinary stocks. The price movements were very similar to those of Indian Government securities, but modified as a result of the surplus profits distributed from time to time as excess dividends.

Mr. HARTLEY WITHERS admitted having felt a certain disappointment on hearing the very careful and able paper the author had submitted. He had hoped to hear how it was that insurance companies were able to manage their investment business so extraordinarily well as they usually did, and he thought some really new principle might be put before the meeting showing how the tremendous strength of the companies had been built up. The author stated that, as a general rule, when the prices of commodities were high the prices of gilt-edged securities were low. That well-known fact, elaborated and worked out in an extraordinarily painstaking manner, seemed to be the net result of the paper. He wished the author had carried his investigations a little further, and taken that long view which, as he said several times, had to be taken by those who conduct the investment business of insurance companies. What was that long view? The author had certainly taken a long view backwards, with very great care and in a very interesting manner, and had shown how the curves he had exhibited agreed, and his paper was an extremely interesting historical document. But, as he himself said, what was wanted was light as

to the future. While it was extremely likely that the prices of commodities would rise a little—perhaps considerably—it was also possible—quite possible, and according to expert opinion probable—that there would be a considerable advance in gilt-edged securities. That, of course, was not inconsistent with what was said in the paper. The author only showed general agreement over long periods, and he called attention to certain lags which defeated the principle as one to be acted upon with invariable success in choosing securities from the point of view of their likelihood to rise or fall. Nevertheless, what everyone wanted to know was how far all the reasons which the author gave as affecting both commodities and securities were likely to be working at the present time.

MR. GEOFFREY MARKS believed that by far the most important part of the practical work of actuaries, if they were connected with life assurance offices, as most of them were, was the management of investments. He was glad to see that there were signs amongst the responsible officers of insurance companies of a very keen interest in the financial problems of their offices. He desired, however, to utter a warning to his fellow actuaries—a warning which he felt it as necessary to give to himself as to them. They must not consider, because they took up, some of them late in life, the very complicated economic questions which had a bearing on the practical problems of finance, that in the short space of business life which remained to some of them and in the few opportunities which a busy life allowed them, they could master the intricacies of those questions. There was a tendency amongst actuaries—and again he applied that to himself—to assume that, because they knew, after a certain amount of study, a good deal about a subject, they knew all about it. He had had that feeling himself at one time with regard to some economic questions, but since, in later years, he had studied them more deeply, he had come to the conclusion that if one could get enough knowledge and experience to judge the value of the theories and opinions of really competent economists, it was as much as one could hope to acquire, and that was perhaps the most useful kind of knowledge that any actuary could have.

MR. A. D. BESANT agreed with Mr. Marks that the investment side of an actuary's work was, from the life insurance point of view, the most important. Like Mr. Hartley Withers, he had been a little disappointed to find that the paper did not seem to lead to any definite conclusions with regard to the future. The author had left out one element of great importance, namely, income tax. The *net* yield on a perpetuity over a good many years before the war and during the earlier stages of the war would be found to be almost constant at somewhere about $3\frac{3}{4}$ per-cent. The more or less sympathetic movements of the author's commodity and Consol curves between 1914 and 1918 were due to Consols having been taken at their nominal yield. If an amended Consol curve were plotted out on the basis of net yield it would be almost flat from 1914 to 1921. One was therefore confronted with the curious

feature that while from 1910 to 1921 commodities had risen rapidly and then fallen slowly, the Consol curve had hardly moved at all. so that again from 1910 to 1921 one seemed to get one of those periods in which, as the author confessed, the value of Consols did not seem to move in accordance with the value of commodities.

The main interest of the paper lay in the fundamental question whether at the present time one ought to invest in short-dated or long-dated securities. He could not help feeling that the time for buying long-dated investments was still a little distant, and that it was speculative rather than safe to invest in long-dated securities to any extent. He shared the author's view that the investments of a life office ought to be more or less balanced between the mortgage class, where capital was intact and interest fluctuated, and the Stock Exchange class, where interest was usually constant and capital value fluctuated. But, in view of financial and industrial conditions, it seemed to him that the danger of income tax, super tax, or corporation tax changes, which would affect the investments of life offices, could not be ignored, and that for safety they ought to confine themselves either to short-dated Government or other securities, where capital would be redeemable at a fixed figure within a short time.

Mr. S. G. WARNER thought that something might be said in justification of the older practice of devoting less attention to investments than was now admitted to be essential. Conditions were different then from what they were now. The field for investment was not nearly so large; the funds of the companies were very much smaller; the whole problem was an easier and a simpler one. And problems of a more strictly actuarial nature had not arrived at the stage of complete solution they had now attained; many of the important investigations, the results of which were now available, were being carried on and many of the ablest actuarial minds were devoting themselves to the solution of the problems which then confronted them.

With regard to the general thesis submitted in the paper, it seemed to him that the author had made out almost too good a case. Practically every variation from the sympathetic movement that the thesis demanded was explained by reference to some outside event of greater or less importance. These events occurred most conveniently to explain away such irregularities as occurred, but were all such events known and taken account of? Could one be quite sure that, if in possession of all the facts, some things might not be found operating in the other direction? The thesis that increased price of commodities meant reviving business and a general desire to invest money in business did not apply altogether to the enormous rise in prices that had recently taken place. That rise was not due to riches, but to poverty, to war conditions, quite exceptional and unprecedented conditions.

Mr. H. E. RAYNES said that the paper concentrated attention on the flow of capital between fixed-interest bearing securities and industry generally. That might have been the predominant feature before the war, but he doubted whether it was now. He thought

that the question for the future was—could industry, as at present organised, supply stocks of capital to add to national savings? If it could, and there were successive years of considerable profits, the rate of interest would decrease; but if industry could not produce the necessary surplus, the rate of interest would remain high. Judging from the difficult conditions now obtaining, one might say industry would have difficulty in producing the necessary surplus, and therefore high rates of interest might be expected to be continued for a considerable period.

Mr. O. T. Falk in closing the discussion, said that the relation between commodity prices and the rate of interest was not, as had been suggested, a relation between high and low prices and high and low rates of interest, but between the change of the rate of value of the monetary standard and the rate of interest. From that point of view, the author's diagram was a little unfortunate; he would have done much better to follow the example of Professor Irving Fisher and trace the relation between the rate of interest and changes in the value of the monetary standard. As several speakers had said, there were other factors of great importance which affected the rate of interest, and for that reason and others—particularly the fact that the period of lag could not be accurately foreseen—he thought the author's rule (if he might so term it) as a guide to investors was of little or no practical use. He had recently made an examination of the price of Consols, practically day by day, from 1874 to 1913, with a view to seeing what might have been done in the way of making profits by changing from short-dated to long-dated securities and *vice versa*. He was inclined to think that even the nimblest and most intelligent investor could not have shifted his securities profitably as often as commodity prices shifted. Such an investor might perhaps have foreseen the rise in the price of Consols from 1874 to 1896, and the big fall from 1897 to 1913. He would have had to be very clever to foresee that rise and fall right at the beginning of the movements, but he might have succeeded after they had progressed two or three years. In addition, he could have foreseen some of the movements in a contrary direction to the big swing movements, such as those that occurred at the time of the Baring crisis, the Boer war, and after the American crisis of 1907. On each of those three occasions, and on one or two others, he might quite well have made a profit of from 5 to 8 per-cent by making a wholesale change from short to long dated securities or *vice versa*, as was required. That, he thought, was the best that could have been done. It was quite certain that such an investor would have had to have in his mind, in order to make those profits, many other things than the author's rule.

It was perhaps by trying to forecast the future, and not by talking about the past, that one tested best the value of theories of this type. With regard to the immediate future, he agreed with Mr. Hartley Withers that it was probable that within the next twelve months or so wholesale prices in this country would rise, and perhaps rise a good deal. They would do that whatever the Treasury and Bank of England might do to stop them rising. With

that rise, there would very possibly occur at first a rise in long-dated securities, exhibiting the usual lag, but he was not at all confident about that. He thought that if it took place at all, it would take place for a very short time, and be followed by a fall in long-dated securities but he was not confident about that either, because his belief was that the security market was not following quite a normal course at the present time.

The outlook with regard to the next big swing probably interested most of those present even more than the outlook of the next few months. There was no doubt that, as regards big movements in interest rates, a change in commodity prices, if pronounced, was a factor of overwhelming importance, and for that reason he believed that, during the next twenty years, the question which was going to dominate the whole financial situation was the gold question. The problem could be divided into two parts. In the past the movement in prices, in so far as gold had affected it, had been largely caused by alterations in the volume of the production of gold. At the present time, however, he did not think the gold production factor was a very important one, because unless something quite exceptional happened such as the discovery of an important new field or the cheapening of the process of production, the effect of new production would be negligible compared with the settlement of the question as to the sufficiency of the present stocks of gold for the support of the present commodity price level. His own feeling was that gold prices, which had now risen from the pre-war level to the extent of something like 40 per-cent, could not possibly be maintained with the present stocks of gold unless some means, not in existence before the war, were discovered of economising gold in the world's financial machinery. What were the possibilities in that direction? Only four occurred to him that were really of any importance. One was that some extraordinary development might take place in the East, that suddenly India or China (to take two examples) might decide to give back to the world the gold which they had been absorbing; the second that the world might decide to dispense with gold altogether; the third that bi-metallism might be resorted to, and the gap in gold filled, in part at least, with silver. All these he considered very improbable. The fourth and most hopeful possibility was that there might be an international development of the gold exchange standard, in the direction not only of dispensing with gold in circulation as coin, but also of economizing in the gold reserves used for settling international indebtedness. Many great authorities thought that development along those lines was quite practicable, but for his own part he doubted it, for the same reasons which led him to doubt the introduction of bi-metallism. Both plans involved international agreements. Any important new step in the direction of economizing in the use of gold was therefore unlikely, and in that case, in the absence of any startling new development on the production side, it seemed probable that there would be a considerable fall in gold prices. With that there would almost certainly be a

rise in the price of securities and a fall in interest rates. The question of gold prices was quite a different question from that of paper currency prices. If on top of what was going to happen to gold prices there occurred a rise in the gold value of depreciated paper currencies caused by attempts on the part of the various countries with depreciated paper to return to pre-war parities, there would be experienced, in many parts of the world, a very heavy fall in commodity prices, a fall which to his mind would be disastrous. He regarded it as very unfortunate that many of the countries in Europe, including England, were aiming at bringing paper prices down to the gold prices of America.

What he had said was highly conjectural, and although he had spoken rather dogmatically he did not want the members to think that he felt that one could speak with certainty about what was going to happen. The times were very difficult, and one had to be ready to revise opinions and policy from hour to hour. In the management of the finances of an insurance company or of any other financial institution, he thought it was unsatisfactory to build too much on inductive inferences. That was one of the greatest troubles that actuaries had to face, for their whole training led them to build up guides to action on the basis of induction. When they came to the field of finance they had got to change their whole mentality. They must go into the market place and get in touch with those who were controlling the various activities of the City, in order to find out quickly what was happening in connection with the factors that affected commodity prices and the rate of interest. If they did not, they would be continually studying past history, and their action with regard to their investments would, generally speaking, be some months or perhaps some years too late.

The PRESIDENT proposed that a hearty vote of thanks be given to Mr. Perry for his paper. He thought everyone would agree that the author had done the Institute a very real service by bringing before its notice a subject which was perhaps the most difficult of all with which the actuary had to deal.

The vote of thanks was carried unanimously.

MR. PERRY, in thanking the members for their resolution, said he realized as much as anyone present the defects of his paper. His object had been to learn, and he wanted to draw the authorities on the subject rather than to teach. Mr. Besant had raised the question as to whether income tax should be deducted and the net yield brought into any diagram of the kind he had exhibited. He (Mr. Perry) did not quite feel that it should. The investor only got the net yield, but everyone else—the man who was earning a salary or wages, for example—had to pay income tax, and they had to compete with foreign markets where the income tax was on a different basis altogether. He thought it was the gross yield which should be the deciding factor. He would like to thank Mr. Falk very much for his remarks. The object he had in view when presenting his paper had been attained more through Mr. Falk's remarks than by anything else.

Another Method of Valuing Policies in Groups. By H. L. Trachtenberg, B.A., A.I.A., *Actuarial Assistant in the Statistical Department of the Medical Research Council.*

I.—Introductory.

All the methods of group valuation submitted to the Institute in the last decade require the calculation of the second sum of the quantities valued, whether in the form of multiplications by $-4\frac{1}{2}$, $-3\frac{1}{2}$, . . . $3\frac{1}{2}$, $4\frac{1}{2}$, *i.e.*, the second sum about the centre, or the more convenient second sum about the end. The object of this paper is to show that the α, β method can be transformed so as to avoid this difficulty.

II.—The Method.

If the quantities in the group valued be linear or of the second degree this transformation takes the simple form of multiplying the total sum assured in the first half of the group by a tabulated γ_1 and the total sum assured in the second half by a tabulated γ_2 and adding. As soon as there is divergence from this linear or parabolic trend, however, an error involving the first difference of the valuation factor is introduced. Thus the circumstances in which the γ formula may be used are sharply defined—the data must be approximately linear or of the second degree. Another formula is deduced, however, which reproduces the α, β result in all circumstances, is intermediate in facility of application between the γ and the α, β formulæ, and still avoids the difficulty referred to in I.

According to this formula the value is obtained by multiplying the total sum assured in the first half of the group by a tabulated α_1 and the total sum assured in the second half by a tabulated α_2, and a simple function of the sums assured by a tabulated β'. The three products are added, thus still replacing the ten products of the detailed valuation by a very much smaller number.

III.—Establishment of the γ Method.

By the α, β method (*J.I.A.*, vol. lii, p. 38), if $f(x)$ be the quantity valued and u_x the valuation factor,

$$\alpha\Sigma(t)f(x)+\beta\Sigma(t)xf(x),$$

where $\Sigma(t)$ stands for $\sum\limits_{-\frac{t-1}{2}}^{\frac{t-1}{2}}$, and $\alpha = \dfrac{\Sigma(t)u_x}{t}$, $\beta = \dfrac{\sum\limits_{\frac{1}{2}}^{\frac{t-1}{2}} u_x - \sum\limits_{-\frac{t-1}{2}}^{-\frac{1}{2}} u_x}{\frac{t^2}{4}}$, gives the value as far as second differences of u_x provided $f(x)$ is linear.

By expressing $f(x)$ in terms of $f(0)$ and its central differences it is seen that as far as second differences in $f(x)$, the second sum about the centre $\Sigma(t)xf(x)$ is identical with

$$\frac{t^2-1}{3t}\left\{\sum_{\frac{1}{2}}^{\frac{t-1}{2}} f(x) - \sum_{-\frac{t-1}{2}}^{-\frac{1}{2}} f(x)\right\}$$

Substituting this expression, and denoting $\sum\limits_{\frac{1}{2}}^{\frac{t-1}{2}}$ by $\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)$, and $\sum\limits_{-\frac{t-1}{2}}^{-\frac{1}{2}}$ by $\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)$, the above formula for the value becomes

$$\alpha\Sigma(t)f(x) + \frac{t^2-1}{3t}\beta\left\{\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x) - \Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)\right\}$$

$$= \left(\alpha - \frac{t^2-1}{3t}\beta\right)\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x) + \left(\alpha + \frac{t^2-1}{3t}\beta\right)\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)$$

$$= \gamma_1 \,.\, \Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x) + \gamma_2 \,.\, \Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)$$

where $$\gamma_1 = \alpha - \frac{t^2-1}{3t}\beta, \quad \gamma_2 = \alpha + \frac{t^2-1}{3t}\beta$$

or $$\gamma_1 = \frac{7t^2-4}{3t^3}\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)u_x - \frac{t^2-4}{3t^3}\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)u_x$$

$$\gamma_2 = \frac{7t^2-4}{3t^3}\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)u_x - \frac{t^2-4}{3t^3}\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)u_x$$

From the above demonstration it is seen that (1) the γ formula gives identical results with the α, β formula as far

as second differences in $f(x)$, (2) like the α, β formula it reproduces the true value as far as second differences of u_x in cases when $f(x)$ is linear.

IV.—Establishment of the α_1, α_2, β' Method.

The substitution of

$$\frac{t^2-1}{3t}\left\{\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)-\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)\right\}$$

for $\Sigma(t)xf(x)$ causes a difference between the γ and α, β formulæ when $f(x)$ is not linear or quadratic. This difference

$$=\left[\frac{t^2-1}{3t}\left\{\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)-\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)\right\}-\Sigma(t)xf(x)\right]\beta$$

which, since (up to second differences of u) $\beta=a_0$, involves the first difference of the valuation factor. Replacing this difference in the γ formula so as to reproduce the value given by the α, β method even in the general case, the form taken by this value is now

$$\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)\,.\,\gamma_1+\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)\,.\,\gamma_2$$

$$-\left[\left(\frac{t^2-1}{3t}-\frac{t-1}{2}\right)\left\{f\left(\frac{t-1}{2}\right)-f\left(-\frac{t-1}{2}\right)\right\}\right.$$

$$+\left(\frac{t^2-1}{3t}-\frac{t-3}{2}\right)\left\{f\left(\frac{t-3}{2}\right)-f\left(-\frac{t-3}{2}\right)\right\}$$

$$\left.+\ldots.+\left(\frac{t^2-1}{3t}-\frac{1}{2}\right)\left\{f\left(\frac{1}{2}\right)-f\left(-\frac{1}{2}\right)\right\}\right]\beta$$

in which as it stands the β multiplier is a function of the f's involving awkward multiplications of the type it is desired to avoid. Re-arrangement, however, will get rid of this difficulty in the case of $t=10$. For the above value may be written :

$$\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)\,.\,\gamma_1+\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)\,.\,\gamma_2$$

$$-\left[\left(\frac{t^2-1}{3t}-\frac{t}{4}-\frac{t-2}{4}\right)\left\{f\left(\frac{t-1}{2}\right)-f\left(-\frac{t-1}{2}\right)\right\}\right.$$

$$+\left(\frac{t^2-1}{3t}-\frac{t}{4}-\frac{t-6}{4}\right)\left\{f\left(\frac{t-3}{2}\right)-f\left(-\frac{t-3}{2}\right)\right\}$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left(\frac{t^2-1}{3t}-\frac{t}{4}+\frac{t-2}{4}\right)\left\{f\left(\frac{1}{2}\right)-f\left(-\frac{1}{2}\right)\right\}\beta$$

in which the multipliers in the β factor now consist of a common portion $\frac{t^2-1}{3t}-\frac{t}{4}$ together with a quantity varying from one end of the group to the other, but such that the values on each side of the centre are equal in magnitude and opposite in sign.

Thus the value

$$=\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)\left\{\gamma_1+\left(\frac{t^2-1}{3t}-\frac{t}{4}\right)\beta\right\}$$

$$+\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right)f(x)\left\{\gamma_2-\left(\frac{t^2-1}{3t}-\frac{t}{4}\right)\beta\right\}$$

$$+\left[\frac{t-2}{4}\left\{f\left(-\frac{t-1}{2}\right)-f\left(\frac{t-1}{2}\right)\right\}+\frac{t-6}{4}\left\{f\left(-\frac{t-3}{2}\right)-f\left(\frac{t-3}{2}\right)\right\}\right.$$

$$\left.+\ \ldots\ldots\ldots\ldots -\frac{t-2}{4}\left\{f\left(-\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right\}\right]\beta'$$

where in the third product β' has been put for $-\beta$.

The coefficient of $\Sigma\left(-\frac{t-1}{2}, -\frac{1}{2}\right)f(x)$

$$=\left(\alpha-\frac{t^2-1}{3t}\beta\right)+\left(\frac{t^2-1}{3t}-\frac{t}{4}\right)\beta$$

$$=\alpha-\frac{t}{4}\beta$$

$$=\Sigma\frac{\left(-\frac{t-1}{2}, -\frac{1}{2}\right)u_x}{t/2}$$

which may be called α_1, and similarly the coefficient of $\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right) f(x)$

$$= \frac{\Sigma\left(\frac{1}{2}, \frac{t-1}{2}\right) u_x}{t/2},$$

which may be called α_2, while the coefficient of β',

$$\frac{t-2}{4}\left\{f\left(-\frac{t-1}{2}\right)-f\left(\frac{t-1}{2}\right)\right\}+\frac{t-6}{4}\left\{f\left(-\frac{t-3}{2}\right)-f\left(\frac{t-3}{2}\right)\right\}$$
$$+\ldots\ldots\ldots-\frac{t-2}{4}\left\{f\left(-\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right\}$$

becomes in the case of $t=10$,

$$2\{f(-4{\cdot}5)-f(4{\cdot}5)\}+\{f(-3{\cdot}5)-f(3{\cdot}5)\}$$
$$-\{f(-1{\cdot}5)-f(1{\cdot}5)\}-2\{f(-{\cdot}5)-f({\cdot}5)\},$$

or, $$2\{f(-4{\cdot}5)-f(-{\cdot}5)\}+\{f(-3{\cdot}5)-f(-1{\cdot}5)\}$$
$$+2\{f({\cdot}5)-f(4{\cdot}5)\}+\{f(1{\cdot}5)-f(3{\cdot}5)\}$$

This is calculated with ease. The data are simply written out cyclically  for the two quinquennial groups, omitting the centre term in each case, and the operations completed as shown:

	$f(-4{\cdot}5)$	$f(-3{\cdot}5)$	$f({\cdot}5)$	$f(1{\cdot}5)$
	$f(-{\cdot}5)$	$f(-1{\cdot}5)$	$f(4{\cdot}5)$	$f(3{\cdot}5)$
Subtract	K_1	K_2	K_3	K_4
	K_3	K_4		
Add	L_1	L_2		
	L_1			
	L_2			
Add	R			

It will be observed that even such a simple multiplier as 2 is dispensed with by repeating L_1, while nothing so cumbrous as a second sum is involved.

As an example of the application of the α_1, α_2, β' method, the complete work of valuing the premiums in group 10 to 19 of the endowment assurances shown in *J.I.A.*, vol. lii, p. 46, is now given:

Valuation of Endowment Assurance Premiums.

Unexpired Term	Premiums
10	6,528·5
11	6,682·5
12	6,845·8
13	7,023·0
14	7,233·3
Total	34,313·1 = α_1 – multiplier
15	5,329·9
16	5,467·7
17	5,627·6
18	5,823·2
19	6,087·5
Total	28,335·9 = α_2 – multiplier

6,528·5	6,682·5	5,329·9	5,467·7
7,233·3	7,023·0	6·087·5	5,823·2
− 704·8	−340·5	−757·6	−355·5
− 757·6	−355·5		
−1,462·4	−696·0		
−1,462·4			
− 696·0			
−3,620·8 = β' – multiplier			

$$\text{Value} = 34{,}313{\cdot}1 \times 9{\cdot}480 + 28{,}335{\cdot}9 \times 12{\cdot}374 + (-3{,}620{\cdot}8) \times (-{\cdot}579)$$
$$= 678{,}013$$

as against the true value 678,141, an error of but −128, or less than ·02 per-cent.

V.—TABULATED VALUES.

In the γ method, for decennial groups,

$$\gamma_1 = {\cdot}232\Sigma(-4{\cdot}5\,,\,-{\cdot}5)u_x - {\cdot}032\Sigma({\cdot}5\,,\,4{\cdot}5)u_x$$
$$\gamma_2 = {\cdot}232\Sigma({\cdot}5\,,\,4{\cdot}5)u_x - {\cdot}032\Sigma(-4{\cdot}5\,,\,-{\cdot}5)u_x$$

and it is only necessary to calculate and tabulate these for contiguous decennia, 10–19, 20–29, &c., not overlapping decennia 10–19, 11–20, &c.

In the α_1, α_2, β' method

$$\alpha_1 = \Sigma(-4{\cdot}5\,,\,-{\cdot}5)u_x \times {\cdot}2$$
$$\alpha_2 = \Sigma({\cdot}5\,,\,4{\cdot}5)u_x \times {\cdot}2$$
$$\beta' = \{\Sigma(-4{\cdot}5\,,\,-{\cdot}5)u_x - \Sigma({\cdot}5\,,\,4{\cdot}5)u_x\} \times {\cdot}04$$

and here again only the values for contiguous decennia are required.

VI.—Conversion Tables and Incomplete Groups.

Although it is easy to formulate rules to deal with the derivation of the A from the a factors, it is easier to tabulate the A's once and for all.

Similarly I think the plan suggested by Mr. Kenchington, *J.I.A.*, vol. lii, pp. 469-70, of confining short methods to the complete groups is most advantageous.

Pension Funds and Income Tax.

[Having regard to the great importance of securing exemption from income tax in connection with existing and future pension schemes, we think it desirable to reproduce in full the section of the Finance Act, 1921, granting exemption under the specified conditions, and the Regulations since made by the Inland Revenue Commissioners.—Eds. *J.I.A.*].

Section 32 of the Finance Act, 1921.

Exemption of Superannuation Funds from Income Tax.

(1) Subject to the provisions of this section and to any regulations made thereunder, exemption from income tax shall be allowed in respect of income derived from investments or deposits of a superannuation fund, and, subject as aforesaid, any sum paid by an employer or employed person by way of contribution towards a superannuation fund shall, in computing profits or gains for the purpose of an assessment to income tax under Case I or Case II of Schedule D. or under Schedule E., be allowed to be deducted as an expense incurred in the year in which the sum is paid :

Provided that :

(*a*) no allowance shall be made under the foregoing provision in respect of any contribution by an employed person which is not an ordinary annual contribution, and where a contribution by an employer is not an ordinary annual contribution, it shall, for the purpose of the foregoing provision, be treated as the Commissioners may direct, either as an expense incurred in the year in which the sum is paid or as an expense to be spread over such period of years as the Commissioners think proper ; and

(*b*) no allowance shall be made under this section in respect of any payments in respect of which relief can be given under section thirty-two of the Income Tax Act, 1918.

(2) Income tax chargeable in respect of an annuity paid out of a superannuation fund to a person residing in the United

Kingdom shall, if the Commissioners so direct, be assessed and charged on the annuitant under Case VI of Schedule D. instead of being deducted and accounted for under Rule 21 of the General Rules, and tax shall be computed on the full amount of the annuity arising in the year of assessment.

(3) For the purposes of this section, the expression "superannuation fund" means, unless the context otherwise requires, a fund which is approved for those purposes, by the Commissioners, and, subject as hereinafter provided, the Commissioners shall not approve any fund unless it is shown to their satisfaction that:

(*a*) The fund is a fund *bona fide* established under irrevocable trusts in connection with some trade or undertaking carried on in the United Kingdom by a person residing therein;

(*b*) the fund has for its sole purpose the provision of annuities for persons employed in the trade or undertaking either on retirement at a specified age or on becoming incapacitated at some earlier age;

(*c*) the employer in the trade or undertaking is a contributor to the fund;

(*d*) the fund is recognised by the employer and employed person in the trade or undertaking.

Provided that the Commissioners may, if they think fit, and subject to such conditions, if any, as they think proper to attach to the approval, approve a fund, or any part of a fund, as a superannuation fund for the purposes of this section:

(i) Notwithstanding that the rules of the fund provide for the return in certain contingencies of contributions paid to the fund; or

(ii) if the main purpose of the fund is the provision of such annuities as aforesaid, notwithstanding that such provision is not its sole purpose; or

(iii) notwithstanding that the trade or undertaking in connection with which the fund is established is carried on only partly in the United Kingdom and by a person not residing therein.

(4) The Commissioners may make regulations generally for the purpose of carrying this section into effect and, in particular, without prejudice to the generality of the foregoing provision, may by such regulations:

(*a*) Provide for the charging of and accounting for tax in respect of contributions (including interest) repaid to a contributor to a superannuation fund and on lump sums paid in commutation of or in lieu of annuities payable out of a superannuation fund as if any sums so repaid or paid were income of the year in which they are repaid or paid;

(*b*) require the trustees or other persons having the management of a superannuation fund, or an employer whose employees contribute to a superannuation fund, to deliver to the Commissioners such information and particulars as the Commissioners may reasonably require for the purposes of this section ;

(*c*) prescribe the manner in which claims for relief under this section are to be made and approved, and in which applications for the approval of a superannuation fund are to be made ;

(*d*) provide for the withdrawal of approval in the case of a fund which ceases to satisfy the requirements of this section ;

(*e*) provide for determining what contributions to a superannuation fund are to be treated as ordinary annual contributions for the purposes of this section.

(5) Where at the commencement of this Act there is in force any arrangement between the Commissioners and the persons having the management of a superannuation fund by which provision is made for allowing any such deductions for the purpose of income tax as may be allowed for that purpose under this section, the arrangement shall, if the fund is approved as a superannuation fund for the purposes of this section for the year 1921–22, be deemed to have ceased to operate as from the 6th day of April, 1921, and, if the fund is not so approved, shall cease to operate as from the 6th day of April, 1922.

(6) In this section the expression "the Commissioners" means the Commissioners of Inland Revenue.

REGULATIONS, *dated* 10 *November* 1921, *made by the Commissioners of Inland Revenue.*

1. Application for the approval of any fund or any part of a fund for any year of assessment shall be made in writing before the end of that year by the trustees of the fund to H.M. Inspector of Taxes for the district in which the office of the fund is situated or the fund is administered and shall be supported by a copy of the instrument under which the fund is established and two copies of the rules and of the accounts of the fund for the last year for which such accounts have been made up and such other information as the Commissioners may reasonably require.

2. Any alteration in the rules, constitution, objects, or conditions of the fund made at any time after the date of the application for approval shall be notified forthwith to the Inspector. In default of such notification, any approval given shall, unless the Commissioner otherwise determine, be deemed to have been withdrawn at the date from which the alteration had effect.

3. If the facts in regard to any fund or part of a fund shall, in the opinion of the Commissioners, cease to warrant the continuance

of their approval under the above-mentioned section then and in such case the Commissioners may withdraw their approval and shall give notice to the trustees of the fund of such withdrawal, and of the date from which such withdrawal is to take effect.

4. If a fund or a part of a fund is approved by the Commissioners the fact shall be communicated to the trustees of the fund in writing and where a fund or part of a fund is approved subject to conditions these conditions shall also be communicated to the trustees of the fund in writing.

5. The expression "ordinary annual contribution" shall mean an annual contribution of a fixed amount or an annual contribution calculated on some definite basis by reference to the earnings, contributions or numbers of the members of the fund.

6. The amount of the employer's contribution which may be deducted as an expense in the case of a local authority which is assessable to income tax in respect of the profits of a trade shall be such part of the authority's total contribution as is made in respect of the persons employed in such trade, and in the case of any other employer shall not exceed the amount contributed by him in respect of the persons employed by him in the trade in respect of the profits of which he is assessable to United Kingdom Income Tax.

7. Where contributions (including interest on contributions, if any) are repaid to the employer the trustees of the fund shall deduct income-tax from the amount so repaid at the rate in force for the year in which the repayment is made and such amount shall be deemed for purposes of income tax, super tax and corporation profits tax to be income of the recipient for that year; the tax so deducted shall be a debt from the trustees of the fund to the Crown and recoverable as such accordingly and the provision contained in Section 2 of the Act of 54 and 55 Victoria, c. 38, in relation to money in the hands of any person for stamp duty shall apply to the tax so deducted.

8. Where any contributions (including interest on contributions, if any) are repaid to an employed person during his lifetime or where a lump sum is paid in commutation of or in lieu of an annuity, income tax on the amount so repaid or paid shall, except in the case of an employed person whose employment was carried on abroad, be paid by the trustees of the fund who shall make such payment at the rate of one-third the standard rate in force for the year in which the repayment or payment is made, and the said tax shall be a debt due from the trustees of the fund to the Crown and recoverable accordingly.

9. Where annuities are paid by a superannuation fund to persons residing out of the United Kingdom, income tax in respect of such annuities shall be deducted and accounted for by the trustees of the fund under Rule 21 of the General Rules applicable to Schedules A, B, C, D and E of the Income Tax Act, 1918 (*a*) provided that income tax shall not be required to be deducted and accounted for in the case of a person so resident whose employment was carried on abroad.

10. The trustees or other persons having the management of a superannuation fund and an employer who contributes to a superannuation fund, shall, when required by notice from the Inspector within fourteen days from the date of such notice :

(*a*) furnish to the Inspector a return containing such particulars of contributions made to the fund as the notice may require ;

(*b*) prepare and deliver to the Inspector a return containing :

(i) the name and place of residence of every person in receipt of an annuity from the fund,

(ii) the amount of the annuity payable to each annuitant,

(iii) particulars of every contribution (including interest on contributions, if any) returned to the employer or to employed persons, and

(iv) particulars of sums paid in commutation or in lieu of annuities.

(*c*) furnish to the Inspector a copy of the accounts of the fund to the last date prior to such notice to which such accounts have been made up together with such other information and particulars as the Commissioners may reasonably require.

11. Where an employer deducts from the emoluments paid to an employed person or pays on his behalf any contribution of that employed person to a superannuation fund he shall include particulars of such deduction or payment in any return of wages and salaries which he is required to furnish under Section 105 of the Income Tax Act, 1918, or the regulations made thereunder.

12. A claim for relief in respect of income derived from the investments or deposits of a superannuation fund shall be made and signed by the trustees of the fund and all the provisions of the Income Tax Acts which relate to claims for any allowance or deduction or the proof to be given with respect to those claims shall apply to claims for relief under Section 32 of the Finance Act, 1921, and the proof to be given with respect to those claims. Any tax for which the trustees of the fund are required to account may be set off in the settlement of such claim.

13. If a fund or part of a fund for any reason ceases to be an approved fund the trustees of the fund shall nevertheless remain liable to account for tax on any sum paid :

(*a*) on account of returned contributions (including interest on contributions, if any), and

(*b*) in commutation of or in lieu of annuities,

in so far as the sum so paid is in respect of contributions made before the fund or part of a fund ceased to be an approved fund, and the provisions of Regulations 7 and 8 shall apply, with the necessary modifications.

LEGAL NOTES.

By Robert Allen Bateman, B.Sc. (Econ.), *Barrister-at-Law.*

In re Morris. Mayhew v. Halton, (1921) 1 Ch. 172, (1922) 1 Ch. 126.

Does a provision for the payment of compound interest with yearly rests operate to capitalize the interest at the end of each year in such a manner as to make it cease to be "interest of money" within the meaning of the Income Tax Acts?

No, said the Court of Appeal affirming a decision of P. O. Lawrence, J. For many years before the death of Morris, who was a lunatic possessed of considerable means, the presumptive next of kin had heavily mortgaged their expectancies with various insurance companies and reversionary interest societies. Two mortgages had been effected with the Norwich Union Life Insurance Society. The first dated 1898 provided that in consideration of the payment by the Norwich Union of the sum of £20,000 the mortgagors conveyed their respective shares and interests in the estate of the lunatic subject to redemption at any time after the death of the lunatic on payment of the sum of £40,000 with compound interest at the rate of 4½ per-cent per annum from the day of the death of the lunatic with yearly rests. The second mortgage and the mortgages of the other companies concerned contained similar provisions.

The lunatic died on 24 January 1915 and soon afterwards an action was commenced to administer the estate and by a judgment of 17 May 1915 the usual accounts were ordered to be taken. By an order of 10 October 1919 the principal sums were paid to the Norwich Union out of funds in Court, and certain other funds sufficient to answer the compound interest were carried over to a separate account entitled "The account "of the Norwich Union Life Assurance and their mortgagors "subject to income tax (if any)." The Norwich Union issued a summons asking for a declaration that the mortgagors were not entitled by way of deduction for income tax to any part of the funds carried over and for payment to the Company of the funds without deduction of income tax except in respect of the period since the funds were set aside.

In his judgment Lord Sterndale, M.R., said: "The question "to be decided is this. On payment off of the mortgage the "mortgagors seek to deduct income tax upon the amount of "the interest from the date of the death of the lunatic, whereas

"it is contended by the mortgagees that the amount cannot be "deducted, but that all that can be deducted is the tax upon "the interest for the last year, if that were paid in the last year, "because it is contended that the meaning of the words "'compound interest' is that at the end of each year, when "the rest is taken, any interest which is overdue at that time at "once becomes capital for all purposes, and therefore when it "is paid to the mortgagees it is a repayment of capital, and not "a payment of interest.

"The question is whether an amount which is equal to "a number of years' interest until then unpaid, is 'interest "'of money, whether yearly, or otherwise, or an annuity, or "'other annual payment', within the meaning of the Income "Tax Act, 1918, Sch. D, Case III, rule 1 (*a*). If it be then income "tax is payable upon it, and if the income tax is payable upon "it, the mortgagors, when the money comes to be paid over, "would be entitled to deduct the tax.

"In my opinion the words 'compound interest' are not "so comprehensive, and do not mean so much as the appellants "contend. The agreement that the mortgagors make—and I "am satisfied that it would probably be the only one that they "ever thought they were making—is that if they do not pay "their interest as it should be paid, after the death of the "lunatic, at its due date, then they will have to pay interest "upon that interest as well as interest upon the capital which "is unpaid. The way in which that is generally done in "keeping accounts is that at the end of the first and every "succeeding year the interest for the year which is unpaid is "added to capital; at the end of the second year the interest "for that year, if it is unpaid, together with interest on the "first year's unpaid interest, is added to the sum made up of "the capital and the first year's interest; and upon those "aggregate sums interest is calculated for the following year "and so on. . . . . This is commonly and conveniently spoken "of as capitalizing the interest. It is capitalizing the interest "in a sense. . . . . It seems to me to be going a very long way "beyond that to say that it is made capital for all purposes, "and that when it is paid, at the expiration of three, four, five "or ten years, it is all paid, with the exception of the last year "as capital. I do not think that the meaning of the words "'compound interest with yearly rests' at all necessarily show, "or indeed do show, that the mortgagors intended that any

"unpaid interest should become capital for all purposes, the "result, no doubt, if it were so, being that they would pay more "than they would if it were to be considered as interest, because "if they had paid their interest at the end of the first year they "would have paid it less tax. That is admitted. If it is to "become capital for all purposes, then they would have to pay "that as part of the capital, without any deduction of tax at "all. I do not believe that such a matter was in contemplation, "and in my opinion that is not the meaning of an agreement "simply to pay compound interest. I think the word "'capitalization' used in many of the books quoted is a "convenient word, but for the purposes for which it has been "used in the argument before us it is a fallacious word, because "it is taken as referring to capitalization for all purposes, "income tax and otherwise. I do not think that is the meaning "of the word. In my opinion when these sums of interest "come to be paid at the end of the time when payment is made, "although interest has been charged upon them, and although "as a matter of book-keeping, they have from time to time "been added to capital, they do not cease to be interest of "money—that is to say, they are overdue interest upon which "interest has been paid."

In the course of his judgment, Warrington, L.J., pointed out that the Court was merely concerned with the terms of the particular document before it and in another case there might be an express provision making unpaid interest capital, but he doubted, without expressing any definite conclusion, whether such a provision could ever achieve such a purpose and whether in any case it would not infringe the general rule which renders void every agreement for the payment of interest in full without allowing deduction for income tax.

Pirie v. Pirie, (1921) S.C. 781

Where a husband after marriage assigns a policy on his life to his wife and after her death and after his re-marriage assigns it to his second wife are the representatives of his first wife entitled to claim the policy monies as part of her estate?

No, said the Court of Session, in the case where the facts were that in 1858 one Pirie had effected a policy on his life with the Rock Life Assurance Company, and on his marriage

in 1861 had made no settlement on his wife. In 1866 he executed and delivered to his wife an assignation of the policy. The deed recited, *inter alia*, that the assignation was made "for sundry onerous causes and considerations", and was "for "her own use and exclusive of my *jus mariti* and right of "administration." The wife died intestate in 1873, being survived by four children. In 1877 Pirie married his second wife who was still living at the time of his death in 1920. In the meantime he had executed a deed of assignation in favour of the second wife in terms similar to the deed of 1866. There was no evidence as to whether any consideration had been given for the assignation in the first case nor whether at the time it was executed the wife possessed any separate estate.

For the representatives of the first wife it was argued that the deed was to be construed literally and the reference to "sundry onerous causes and considerations" was sufficient to raise the presumption that such existed in fact. For the second wife it was argued that as donation was not suggested the only alternatives for the grant of the assignation were provision or the discharge of debts. The assignation satisfied the three conditions of a provision for the wife contingent on her survivance—(1) there was no antenuptial marriage contract, (2) the husband was solvent, and (3) the provision was to take effect at the dissolution of the marriage. If it had been a question of the discharge of debts the deed would have been more specific.

In giving judgment, the Lord President, Lord Clyde, said that "the question whether the first wife's representatives are "entitled to the proceeds of the policy depends on their being "able to make out some right in her under the assignation in "her favour higher than that of a wife in whose favour a "postnuptial provision is made. It is, in short, for them to "show that she was an assignee who acquired right to the "policy by some onerous transaction (other than marriage) "with her husband." After pointing out that in the case of *Galloway* v. *Craig* the question was as between donation and provision, he continued: "in the present case the alternatives "are between provision and onerous transaction. What "evidence is there in favour of the latter as against the former? "I think that much must turn on the actual words used in the "assignation to describe the cause of granting. Indeed these "appear to be the only available evidence on which the onerous

"character of the transaction can be supported. The "assignation recites 'sundry onerous causes and considerations.' "What, then, is the significance of that? It appears to me "that, so long as such a recital can be satisfied by the duty and "obligation which rest on a husband to provide for his wife, "it is illegitimate to attribute to these words any wider "meaning."

There was a further point that the assignation had been duly delivered to the first wife and in law she would have been entitled to the possession of the policy itself and could have sold it or surrendered it. In regard to a sale she could have conferred no higher right to the policy than she had. In regard to surrender it was doubted if an effectual surrender could have been made without the consent of the husband.

In re Clancarty, (1921) 2 I.R. 377.

Is the equity of redemption in a life assurance policy "goods or chattels" within the meaning of the Irish Bankrupt and Insolvent Act, 1857 (20 & 21 *Vict. c.* 60) *s.* 313*?*

Yes, said the Irish Court of Appeal, affirming a decision of Pim, J. The facts of the case were that in 1905 *The National Mutual Life Assurance Society* granted to the Earl of Clancarty two policies on his own life. On the same day he mortgaged them together with certain lands to the Company and the latter made a note of the mortgage and retained the policies. In 1907 the Earl mortgaged the policies with the lands to Duncan Blair. Later in 1907 the Earl was adjudicated bankrupt and notice was given to the insurance company by the assignee in bankruptcy. In 1915 the lands comprised in the mortgage having been sold the insurance company was paid out of the proceeds of the sale and one policy was handed to the assignee in bankruptcy. In 1920 this policy was surrendered to the company. Towards the end of 1920 Blair gave notice of his claim to the assignee. Then an order was made by the Court for the sale of the policy. The policy having already been surrendered, the surrender value was paid into Court. Blair next gave notice to the insurance company of his mortgage. In the present action he claimed to be paid out of the surrender value proceeds the amount due to him. Pim, J.,

giving judgment said : " There was in the bankrupt at the time " of his adjudication an equity of redemption in the policies, which, " even at that time, had a very considerable value. This equity " of redemption, to which the applicant (*i.e.*, Blair) had a legal " right, remained in the order and disposition of the bankrupt, " inasmuch as the applicant had given no notice of his mortgage. " . . . . . Now, what is an equity of redemption ? . . . . . One " thing is quite clear from the decisions namely, that an equity " of redemption, whether it be called ' an estate ', or ' an " ' interest ' or an ' equitable right ', is a real thing of the nature " of property which will vest in the assignee. . . . . Now, this " ' interest ', or ' equitable right ', or ' estate ', was in the " bankrupt in this case at the date of his bankruptcy. If the " applicant here had given notice under sect. 3 of the Policies " of Assurance Act, 1867, this equity of redemption, by force " of that notice, would no longer have been in the order and " disposition of the bankrupt, but would have become the " property of the second mortgagee, leaving in the bankrupt " another equity of redemption subject to the two earlier " mortgages.

" I have come to the conclusion that this equity of " redemption in the policies of insurance is an interest or " equitable estate in the said policies, and comes within the " meaning of the words ' goods or chattels ' in the 313th section. " . . . . I hold that it vested in the assignee at the date of the " bankruptcy. I further hold that it was in the possession of " the bankrupt at the date of his bankruptcy, with the consent " of the true owner, the present applicant. That being so, the " application fails."

There is a further point of interest in the case because it had been argued that if sect. 313 did not cover the case, the claim was barred by the Statute of Limitations. Pim, J., was of opinion that the Statute did not apply as its effect had been stopped by the bankruptcy proceedings and further " there is " no Statute of Limitations dealing with policies of insurance."

In confirming the views of Pim, J., the Court of Appeal had apparently nothing to add.

Wakefield v. Whiteway, Laidlaw & Co. (1922) 1 Ch. 200.

Where a company has paid both British and Colonial income taxes and a repayment has been obtained under section 43 *of the Finance Act of* 1916 *are preference shareholders, who are entitled to a fixed rate of interest from which the full British rate of income tax has been deducted, entitled to share in such repayment?*

This case is only referred to as bringing a little more up-to-date the decision in *Scottish Union and National Insurance Company* v. *New Zealand and Australian Land Company*, which was reported in full in the last issue of the *Journal*. It will be recalled that the House of Lords answered the above question in the negative, thus setting at rest the doubt which had existed since the decision in *Rover* v. *South African Breweries*. The present case was brought in view of the alteration in the law contained in the Finance Act, 1920, s. 27, sub.-s. 5, which reads, "Where under Rule 20 of the General Rules applicable "to Schedules A, B, C, D and E, a body of persons is entitled to "deduct income tax from any dividends, tax shall not in any "case be deducted at a rate exceeding the rate of the United "Kingdom income tax as reduced by any relief from that tax "given under this section in respect of any payment of Dominion "income tax." It was suggested in argument that the language of this sub-section in itself showed that it had been enacted with the object of repealing the decision in the *Scottish Union* case, more particularly in view of the contrary decision in the lower Court in the case of *Rover* v. *South African Breweries*. Sargent, J., in dismissing the action said that he could "see "no reason for attributing the enactment of sub-s. 5 to a desire "on the part of the Legislature to overrule or modify a legal "decision which appears to me a satisfactory one from every "standpoint."

REVIEWS.

A Treatise on Probability. By JOHN MAYNARD KEYNES, *Fellow of King's College, Cambridge.*

[Pp. xi + 466. Macmillan & Co., London, 1921. Price 18s. net.]

THE study of statistical frequency has arisen out of the study of probability: and it still retains a good deal of its terminology, though it has very little to do with it. When, therefore, we come

across a book on Probability, we expect that it will have some bearing on the right use of statistics.

This is a large (and expensive) book, and its contents are varied. It deals, in part, with matters which are of interest to the inductive statistician. But will it be of any help to him ? It is a difficult question to answer, not merely because the author does not seem to be able to get at the statistician's point of view, but also because the statistician will not find it easy to understand the author.

The difficulty strikes us at the very beginning of the book. It is necessary to read some way in order to understand even the first paragraph, which states that the theory of probability treats of the different degrees in which knowledge obtained indirectly, *i.e.* by argument, is conclusive or inconclusive. The truth is that the book is not a treatise on probability, but a treatise—if the word is permissible—on the logic of probability; as the author puts it on p. 98, it is concerned with the general theory of arguments from premisses leading to conclusions which are reasonable but not certain.

The reader who is accustomed to translate questions of probability into the language of frequency will find the book rather exasperating. Even in the earlier pages the author, as soon as he gives examples by way of illustration, seems to confuse two things which are widely different but have the same general name—direct probability, or the probability that a named certain (or practically certain) future event will be accompanied by a named doubtful event, and inverse probability, or the probability that a named past event was accompanied by a named doubtful event. In practically all cases of the former kind the idea of probability implies the idea of betting, and, although there will be individual divergences of opinion, there is something approaching an objective probability. In the other class of cases, so far as they are not purely theoretical, a numerical statement of probability is meaningless. Mr. Keynes deals with the measurement of probabilities in his third chapter. After giving some examples of cases in which estimates of probability have actually been made, he says, in reference to the question of ascribing a particular picture to a particular painter :

> " We can say that the presence of certain peculiar marks in a picture increases the probability that the artist of whom those marks are known to be characteristic painted it, but we cannot say that the presence of these marks makes it two or three or any other number of times more probable than it would have been without them. . . . . ."

This is a question of inverse probability. He continues, in the next paragraph :

> " Or consider the ordinary circumstances of life. We are out for a walk—what is the probability that we shall reach home alive ? "

This is a question of direct probability. There is no mention, however, of the fact that the two things are quite different.

It is not until nearly a quarter of the way through the book that he considers what he calls " the frequency theory of

probability", mainly by discussion of Venn's *Logic of Chance*. But his descriptions are inconsistent. On p. 96 he says that

> "The science of probability is, according to [Venn], *no more* than a method which enables us to express in a convenient form statistical statements of frequency."

This is fairly correct. But on the next page he refers to

> "the main grounds which have induced Venn to regard judgments based on statistical frequency as the only cases of probability which possess logical importance."

He does not seem to realize that a statement as to frequency—a statement that, under normal conditions, about 5 per-cent of persons of age x die within the next twelve months—is a different thing from a judgment, based on such a fact, that, if B is of age x, his probability of dying within the next year is $\frac{1}{20}$—a "judgment" to which it is difficult to attach any precise meaning. On p. 98 he says, apparently by way of concession, "We may call a statistical frequency a probability, if we choose"; the answer is that we don't choose—it is the very thing we are trying to avoid.

Part I, in reference to which the above remarks are made, deals with Fundamental Ideas. Part II is on Fundamental Theorems. But it is difficult to recognize these as theorems in probability, as we understand it: they are theorems as to the degree of rational belief in propositions. The consideration of these theorems is treated as a branch of logic, and a special notation is adopted, which has the merit of brevity but the demerit that it does not follow the ordinary laws of algebra.

It is possible that a good deal of ordinary reasoning as to statistical frequency is loose: and criticisms such as those made by Mr. Keynes may be useful. He points out, for instance, the importance of the assumption of "independence" in the multiplication-theorem—the theorem that, if the probabilities of two events separately are p_1 and p_2, the probability of both together is $p_1 p_2$. But, even here, his criticisms relate to statements which are not entirely modern: one has a suspicion, as in some other cases, that he is flogging dead horses.

It is even doubtful whether his own method is severely logical. For his definition (on p. 138) of "independence" seems to require two conditions, whereas one only would be sufficient.

The title of the last chapter of this Part, "Some Problems in "Inverse Probability, including Averages", suggests something interesting. The interest of the first problem—the second is merely an extension of it—is purely historical. The problem, published by Boole in 1851, may be stated concisely as follows:

E cannot happen unless A_1 or A_2 happens. The total probability of A_f ($f = 1$ or 2) is c_f. If A_f happens, the probability of E is p_f. Find the total probability of E. Mr. Keynes says that Cayley gave a solution, which Boole declared to be erroneous; apparently Cayley's terms were so ambiguous that it is impossible to say whether he was right or not. The interesting

thing is that so recently as 70 years ago eminent mathematicians could be attempting to solve a problem for which the data are insufficient. If we look at the matter from the frequency point of view, drawing a double-entry table for A_1 and A_2 present or absent, we see this at once. We see that c_1, c_2, p_1, p_2 are not involved separately, but only in the forms c_1p_1 and c_2p_2; and we see that we require one more numerical quantity of the same kind. The problem, in fact, is a catch; and the struggling reader may well complain that the author nowhere states this.

But two points of interest arise out of this problem. The first is that the difficulty is just the sort of difficulty which from the frequency point of view can be detected at once. Even recent writers on the subject of statistical frequency do not seem fully to realize that it has long ceased to be a mere theory of probability.

The second point is as to system of notation. Mr. Keynes says at the beginning of this chapter that the problems are problems in logic and that the chapter is introduced in order to show the power of his method and its advantage in ease and especially in accuracy. Now, the method being one of logic rather than of mathematics, one thing that it ought definitely to do is to show up, automatically, any inconsistency, superfluity, or insufficiency in the data. But we cannot see that it does this. In the fourth line of this proof the author says "let $a_1a_2/eh = z$", without giving any reason at all for the introduction of z.

This problem, then, is not very well treated. On the other hand, the discussion—reprinted from the *Journal of the Royal Statistical Society* for February 1911—of the various kinds of "mean" is of distinct value to the mathematician. Other points, such as the "method of least squares", are only mentioned briefly. In reference to the arithmetic mean, Mr. Keynes trots out the old objection that the law of distribution of $f(x)$ is not the same as that of x; but he does not consider the rather important question whether an asymmetry produced by change of variable is essentially of the same kind as, or different from, the general asymmetry which is found to hold in certain particular classes of phenomena.

Passing over Parts III and IV, which deal respectively with "Induction and Analogy" and with Some Philosophical Applications, we come to Part V, on the Foundations of Statistical Inference. This is the part that one would expect to appeal to the ordinary reader. But it is disappointing. Various matters—isolated theorems, or extracts from individual authors—are discussed and criticized; but the discussions and criticisms do not lead to much, and there is always the difficulty of agreeing with the author's way of looking at the matter.

In reference to Bernoulli's Theorem, for instance—the theorem that if the probability of E accompanying A is p then on m occasions of A (m being large) the most probable number of occurrences of E (*i.e.* the number of occurrences for which the probability is greatest) is mp—he says (p. 341):

"Consider the case of a coin of which it is given that the two faces are either both heads or both tails: at every toss, provided that the results of the other tosses are unknown, the probability of heads is $\frac{1}{2}$ and the probability of tails is $\frac{1}{2}$. . . . . ."

This can only be true of probability based on ignorance. If a bag contains an even number of coins, all alike to the touch, half being both heads and half being both tails, there is no harm in saying that the probability of drawing a coin which is both heads is $\frac{1}{2}$: but this is a very different thing from saying that when a coin has been drawn the probability that it is both heads is $\frac{1}{2}$. It must be confessed that Mr. Keynes is able to produce a good many statements, by writers of eminence, which give him a basis for adverse criticism, and possibly on this point he would be able to quote some of them on his side. But the case illustrates the difficulty of finding any common basis to start from.

In the chapter which deals with Bernoulli's theorem he mentions (p. 353) a theorem by Tchebycheff, which is not very well known. The algebraical treatment occupies two pages, but it can be stated briefly as follows. Let x, y, z . . . be n independent variables whose means are a, b, c . . . , deviations from means ξ, η, ζ . . . (so that $\xi \equiv x - a$, &c.), and mean squares of deviation from means α^2, β^2, γ^2 . . . (It is not necessary that the number of possible values of each variable should be infinite or that the number should be the same for all variables.) Tchebycheff first shows that the mean square of $\xi + \eta + \zeta + \ldots$—*i.e.*, the mean value of $(x + y + z + \ldots - a - b - c - \ldots)^2$, every value of x being taken with every value of y and every value of z, &c.—is $\alpha^2 + \beta^2 + \gamma^2 + \ldots$ (This is now well known.) Call this D^2. Then the special theorem is that the proportion of cases in which $\sigma \equiv \xi + \eta + \zeta + \ldots$ is numerically greater than θD, where θ is any ratio > 1, is not greater than $1/\theta^2$. [For, if this were not so, then, in finding the value of D^2 for the total N values of σ, the part contributed to ND^2 by these particular cases would be greater than $N/\theta^2 . \theta^2 D^2$, *i.e.* than ND^2; which is impossible.]

The last chapter gives an "outline of a constructive theory", which we cannot profess to be able to summarize. The book concludes with a useful bibliography—unfortunately not classified but arranged under names of authors alphabetically—and an index.

It will be seen that, as stated at the beginning of this notice, the contents are varied. Our general conclusion, as regards matters of interest to readers of this *Journal*, might be somewhat as follows. The theory of probability and the theory of statistics are so closely related that, while some writers have treated probability as based on statistical frequency, others, primarily interested in statistics, have made the mistake of treating it as a branch of probability. Mr. Keynes is dissatisfied with the existing views as to probability, and finds it necessary to attack them, in order to establish the theory on a firm foundation. On his definition of "probability", the attack should extend to the theory of statistics. But he experiences some difficulty in finding a position on which to concentrate the attack. He does succeed in throwing doubt on

the validity of certain theorems which some of us, if we had known of them, might have regarded as mathematical curiosities; and he also finds passages in individual memoirs which show that the writers have been too hasty in accepting principles or drawing conclusions. But it can hardly be said that he shakes the foundations; and he apparently does not claim to have made any substantial contribution to reconstruction.

W. F. S.

Medical Examination for Life Insurance. By THOMAS D. LISTER, C.B.E., M.D., F.R.C.S., M.R.C.P.

[Pp. 168. London, Edward Arnold & Co., 1921. Price 10*s*. 6*d*. net.]

THE preface to this book explains that the author "has attempted "to conceive the difficulties of the insurance examiner faced with "a blank form to fill up to the best of his ability in a short time", and the book itself is dedicated "To those who know their medical "work and may have to apply it to insurance."

Such a book should be welcome to the many medical men up and down the country who are called on from time to time to examine for life assurance and Dr. Lister is well qualified to write on his subject, being Medical Officer to three important insurance Offices and President of the Assurance Medical Society.

The attention of an actuarial reviewer is naturally attracted in the first instance by Dr. Lister's references—although relatively slight in extent and importance—to the actuarial aspects of his subject. We cannot say that these add, in our view, to the value of the book. We doubt, for instance, whether the discussion of sources of profit (p. 19) and the attempt to explain what extra mortality is involved in a given rating-up (p. 159) are likely to convey much enlightenment to Dr. Lister's readers.

In turning to the medical aspect of the book—which is, of course, its main aspect—we feel that we are treading on dangerous ground, and are not forgetful of the old adage that "a little knowledge is a dangerous thing."

The author's general advice to the examiner is excellent and we fully endorse his opinion that what is wanted above all is a careful examination and a precise statement of the facts with full answers to all questions put. As a rule the facts speak for themselves and nothing is gained by asking the examiner to interpret the facts and suggest a suitable rating if the case is not an average one. It is the duty of the chief medical officer to interpret the facts, and in consultation with the actuary to assess the risk.

The author states that "as a matter of actual practice" risks are assessed "on lines that are apparently almost entirely empirical", but we do not think that other chief medical officers will agree with this suggestion. Risks are assessed in the light of experience and the limited statistical information available, and we are still far from finality in the assessment, but that is not empiricism.

The help of the actuary is not infrequently essential in the assessment of special risks, and Dr. Lister would probably agree that it is given (so far as data admit) in a scientific spirit—notwithstanding the suggestion on p. 160 (not, perhaps, intended to be taken too seriously), that actuaries sometimes use fallacious arguments as a "blind" to their medical officers. The paragraph in which this suggestion occurs is particularly interesting because the author's own argument is open to criticism. He asserts that "assuming that there is an extra risk in the proposal, it is clear that an extra premium must be much larger for a limited payment policy, the extra being only payable for the term of years selected, than for a policy for which premiums are paid throughout the whole course of the insurance." In fact it happens not infrequently that the extra premium equivalent to an addition of so many years to the age calculated on the whole life table is greater than the corresponding extra premium calculated on the limited payment table, as may be readily seen by examining the $O^{[M]}$ tables of net premiums.

Dr. Lister explains very clearly the many special points to be considered in assessing the various types of special risks and indicates the various ways in which they can be dealt with in practice. In more than one instance, however, he suggests that the risk of death from some particular disability or disease should be excluded altogether from the policy as an alternative to the payment of an extra premium. While it is true that in exceptional cases policies are sometimes issued excluding death from some special cause, these policies are never satisfactory either to the policyholder or to the insurance company and we do not think that many insurance managers would advocate such an arrangement.

In some respects Dr. Lister's views would not meet with general acceptance amongst his colleagues in the medical profession, but these points are still largely a matter of personal opinion and ought not perhaps to be regarded as blemishes. Some instances may be mentioned. Dr. Lister speaks of the athlete who presents a moderate degree of enlargement of the heart due to hypertrophy from muscular exercise, but we believe that some modern physicians do not now admit the existence of what used to be called the hypertrophied form of athlete's heart. Again, Dr. Lister gives the rough and ready rule which is fairly widely used for the estimation of the normal blood pressure, this rule being to add the proposer's age to 100. Some authorities, however, including those who have studied this matter most closely, do not admit that increase in age should normally be accompanied by an increased blood pressure.

Many chief medical officers and insurance managers would feel that Dr. Lister is inclined to take too lenient a view in connection with proposers who have suffered from syphilis. He suggests that in some circumstances after the lapse of a considerable number of years from the attack without any untoward results manifesting themselves the risk can be regarded as a normal one. The valuable medico-actuarial investigation made in America and referred to in *J.I.A.*, vol. xlviii, p. 188, gave the results of American

experience and these certainly do not confirm Dr. Lister's views. It is interesting to note too that he does not comment on the particular nature of the treatment and makes no mention either of the modern Salvarsan treatment and the results obtained from it or the Wassermann blood test which is now in common use. Dr. Lister will, however, probably agree that the Salvarsan treatment has not been sufficiently long in use to justify any definite deductions as to its effects on the later manifestations of the disease, and also that a negative Wassermann test does not exclude the possibility of a recurrence of the symptoms of syphilis.

H. E. M.

Tracts for Computers. Edited by KARL PEARSON, F.R.S. I.—*The Digamma and Trigamma Functions. By* ELEANOR PAIRMAN, M.A. II. *and* III.—*On the Construction of Tables and on Interpolation. By the* EDITOR. IV.—*Facsimile Re-issue of Legendre's* Γ-*Function Tables.* V.—*Table of the Coefficients of Everett's Central Difference Interpolation Formula. By* A. J. THOMPSON, B.Sc. VI.—*Smoothing. By* E. C. RHODES, B.A.

[Cambridge University Press, 1920–1921. No. I 3*s*. net, Nos. II–VI 3*s*. 9*d*. net each.]

THE series of tracts which is being issued by the Computing Section of the Department of Applied Statistics of University College, London, under the general title of *Tracts for Computers* is a notable addition to the equipment of the statistical laboratory. Both in actual table construction and in research it represents an expenditure of time and thought that calls for grateful acknowledgment. It gives the impression of being not in any sense task-work but rather the product of a well-conceived idea taken up with enthusiasm and ready co-operation by the Editor's staff: it is the outcome of much practical experience in computation; and the tracts succeed each other with commendable rapidity. In the last-mentioned respect indeed the series may be regarded as an exemplification of Carlyle's maxim that "in all things, writing or other, which a man engages "in, there is the indispensablest beauty in knowing how to get "done." It certainly "gets done." At the time of writing the six tracts mentioned in the heading of this notice had already appeared, and others—including one on quadrature methods—were in course of publication or preparation.

The two tracts by the Editor, although not first in order of publication, may presumably be regarded as laying down the general lines on which the actual work of table-construction—which, as indicated by Tracts I, IV and V, is apparently intended to form an important part of the plan of the series—is to proceed. Prof. Pearson holds that, although the introduction of calculating machines has in some instances superseded the use of tables, "the

coming age will be one of many tables"—not so much, however, tables of the old sort, but tables made to save great occasional labour, that is to say, tables of complicated functions of which the calculation direct from their formulas would require "some hard thought and hard computing." But exigencies of economy and space even as regards single variable functions—and still more as regards functions of two or more variables—preclude the publication of numerous tables which can be entered directly or by first differences only. There is also the computer to be considered. It is necessary then to find some compromise between the conflicting requirements of computer, printer and table-user. Prof. Pearson's conclusion is that the tables of the future must be tables with relatively wide argument-intervals (not necessarily the same in all parts of a table) and with one or more tabulated even central differences, entailing on the part of the user some knowledge of interpolation and a little more trouble than is required by the old first difference table. "Ten tables of different functions with two "central difference entries are of far greater value to the computer "than a single function table to a tenth of the interval, and so "of the same bulk, which can be entered with a single forward "difference." With the necessary qualification that the functions must be of more or less equal importance the doctrine seems sound in substance as well as arresting in form, but its application to actuarial tables may perhaps be found in the direction rather of supplementing than of recasting our present type of table. In the case of a standard experience extensively used in practical work complete tables of certain functions (of $a_{x\overline{t}|}$, for example, for all values of x and t) seem indispensable, but they might possibly be supplemented with advantage by wide-interval tables of some of the two or more life functions that are occasionally but not very frequently needed. A table of A^1_{xy}, for example, for decennial values of x and y would admit of the value of the function for any combination of ages being calculated with very little trouble and probably quite as accurately as it can be obtained by means of the ordinary formula depending (practically) on the difference of two nearest third place annuity-values. A simple first central difference interpolation based on $A^1_{30\,60}$, $A^1_{40\,60}$, $A^1_{50\,60}$, $A^1_{40\,50}$, $A^1_{40\,70}$ gives the Carlisle 3 per-cent value of $A^1_{41\,62}$ as ·14796—the tabulated value being ·14841. A remarkable illustration of the possibilities of a wide-interval table with a tabulated second central difference is given by Prof. Pearson at the end of Tract II in the form of a one-page 7-figure log and antilog table by means of which the log or antilog of any 7-figure number can be found to 7 places.

The greater part of Tract II is devoted to the derivation of convenient formulas for completing a mathematical table by interpolation after a skeleton has been constructed by direct calculation or quadrature and for entering a table which has already been constructed—"mid-panel" formulas for general use throughout the greater part of the table, "mid-point" formulas for use in the immediate neighbourhood of a calculated value, "side-panel"

formulas for the difficult regions at the edges of a table, and "bridging formulas" for dealing with regions where the argument intervals change. Most of this work is designed to secure a degree of accuracy which would have no meaning as a rule for actuarial purposes—the coefficients in many of the formulas being given to 10, and in some even to 12, places of decimals—but it is all very instructive, and the artifices used for "bridging" and at the boundaries of a table might occasionally be of practical use to the actuary. Prof. Pearson prefers formulas of a Lagrangian type for purposes of construction, but recommends for use in entering a table Everett's central difference ("mid-panel") formula or an analogous "mid-point" formula in u_{-1}, u_0, u_1 and their 2nd, 4th. &c., central differences—the latter involving an odd instead of an even number of u's and thus giving the advantage, in the immediate neighbourhood of a calculated value, of the ordinary central difference formula without requiring the use of odd central differences. We note one other point only in this Tract that seems to call for comment. Prof. Pearson states—in an incidental reference to the subject of inverse interpolation—that the problem of finding the value of x corresponding to a given value of u_x when differences above the second have to be taken into account "would involve at least the solution of a cubic for the required "value of the argument." This appears to be assumed also by Mr. A. J. Thompson in a more formal discussion of the subject in the preface to Tract V (pp. xii–xv), but it overlooks the fact that by interchanging x and u, *i.e*, by regarding u as the variable and x as the function, we can, either by a Lagrange formula or by divided differences, obtain an interpolated value of x for a given value of u_x based on any desired number of u's. (*Cf. Text-book*, Part I, New Edition, revised, p. 206).

In Tract III* we enter on less familiar ground. Here again much of the work aims at an accuracy which is far beyond any actuarial requirements, but it is also in the main pioneer work in principles as well as in their application, and this renders it of great interest. Very little has been published hitherto on the subject of two-variable interpolation—mainly no doubt because the need for it has not arisen. For ordinary purposes the rough process of interpolating first for one variable and then (from the resulting values) for the other and the more accurate second forward difference formula have been found sufficient. It may be admitted that neither of these methods would be satisfactory if it were necessary to take into account high orders of differences. Prof. Pearson objects to the first because it is laborious and because the results obtained by interpolating in the respective orders xy and yx will not necessarily be in accordance—which does not seem, however,

* This tract is dedicated "to my fellow members of the Actuaries' Club", with the following extract from a letter written in 1675 by John Evelyn, F.R.S., to John Aubrey, F.R.S.: "Sir, I beseech you to accept or pardon these trifling "interpolations, which I have presumed to send you." We venture to congratulate Prof. Pearson on the recollection or discovery of this truly delectable quotation.

to be the fact (a footnote in the tract notwithstanding) when only first differences are used—and to the second because it does not make use of the nearest tabulated values. The method suggested by Elderton in *Biometrika*, vol. vi, and referred to in this *Journal*, vol. xl, p. 299, of interpolating between the values lying on the line passing through 0, 0 and x, y (*e.g.*, for $a_{41\,.\,53}$ between $a_{35\,.\,35}$, $a_{40\,.\,50}$ and $a_{45\,.\,65}$) answers fairly well in two life tables where the values are tabulated for quinquennial ages and interpolated values are wanted for integral ages only, but Prof. Pearson suggests that difficulties would arise in the case of a "precipitous" table. What is wanted is a formula depending on the tabulated values nearest to the required value, but this cannot be obtained in general by the process of passing a surface through the selected points in the same simple way that a curve can be passed through the m nearest points in single variable interpolation. A surface can be passed through the 6 or 10 points respectively involved in a second or third forward difference interpolation—the forward difference formula in effect gives its equation—but not as a rule through the 9 points (for example) of which 0, 0 is the mid-point. Prof. Pearson finds a solution of the problem in an extension of central difference interpolation to two variables. To take a simple example, the ordinary central difference formula, taken to second differences, gives (in Sheppard's notation)

$$u_{xy} = \left(1 + x\mu\delta + y\mu\delta' + \frac{1}{2}x^2\delta^2 + \frac{1}{2}y^2\delta'^2 + xy\mu\delta\mu\delta'\right)u_{00}$$

—a surface of the second degree passing through the five points u_{00}, $u_{-1,0}$, $u_{1,0}$, $u_{0,-1}$, $u_{0,1}$ and very near the four additional points $u_{\pm 1,\pm 1}$. The ordinary central difference formula is, however, unsuitable for table-construction on account of its requiring the tabulation of odd as well as even central differences; besides it gives interpolated values round a central point, not—as is often more convenient—round a central space. The plan adopted in the Tract is to extend to two variables the "mid-panel" and "mid-point" formulas of Tract II. This is a rather more complicated process, and introduces a little uncertainty as to what order of differences particular terms or combinations of terms represent, but it gives formulas of both the types mentioned above and involves even differences only. The "mid-panel" formula leads generally to formulas involving the tabulated values in successive rings concentric about the point $\frac{1}{2}$, $\frac{1}{2}$—up to second differences, for instance, it gives a quartic surface (with the x^4, x^2y^2 and y^4 terms omitted) passing through $u_{0,0}$, $u_{0,1}$, $u_{1,1}$, $u_{1,0}$, and the 8 terms forming the next outer ring—the "mid-point" to formulas in u_{00} and the successive rings concentric about the point 0, 0 together with some outlying but symmetrically arranged values. An interesting alternative plan, called by Prof. Pearson a "semi-Lagrangian" process, of obtaining a formula depending on n nearest points is to pass a surface of order m through p of the points ($n > m > p$) and to determine the remaining

$m-p$ constants by least squares to give the best fit to the remaining $n-p$ points. All this of course contemplates the use of relatively high orders of differences. For ordinary actuarial purposes we are inclined to think that the most useful formulas for interpolation in the space for which x and y are both positive and $<\frac{1}{2}$ (which includes all cases if the origin and direction of axes are suitably chosen) are the 5-point formula

$$u_{00} + \frac{1}{2}x(u_{1,0} - u_{-1,0}) + \frac{1}{2}y(u_{0,1} - u_{0,-1})$$

and the 6-point formula consisting of the same terms with the addition of

$$\frac{1}{2}x^2(u_{1,0} - 2u_{0,0} + u_{-1,0}) + \frac{1}{2}y^2(u_{0,1} - 2u_{0,0} + u_{0,-1}) \\ + xy(u_{1,1} + u_{0,0} - u_{1,0} - u_{0,1})$$

The former is the ordinary first central difference formula and also the simplest member of the "mid-point" family, while the latter (which has the merit of involving 6 points through which a surface of the second order can be passed) is of the same type as the 21-point formula mentioned on page 5 of the Tract as having been suggested by Mr. H. E. Soper. It may be of interest to give the results obtained by these and other simple formulas for $u_{\frac{1}{2},\frac{1}{2}}$ in the example given in the Tract, the true value being ·8851140; the example is favourable to the line formulas, as it is possible to use relatively near values, and unfavourable to the 5-point formula, as $\frac{1}{2}$, $\frac{1}{2}$ is as far as possible from 0, 0:

Nature of Interpolation	Values used	Result
Line	$u_{-1,-1}$, $u_{0,0}$, $u_{1,1}$	·8851358
Do.	Do. and $u_{2,2}$	·8851133
First difference, for x and y successively	$u_{0,0}$, $u_{0,1}$, $u_{1,1}$, $u_{1,0}$	·8842797
First central difference ...	$u_{0,0}$, $u_{1,0}$, $u_{-1,0}$, $u_{0,1}$, $u_{0,-1}$	·8856878
2nd order surface	Do. and $u_{1,1}$	·8851610
Do. forward difference ...	$u_{0,0}$, $u_{0,1}$, $u_{0,2}$, $u_{1,0}$, $u_{2,0}$, $u_{1,1}$	·8950406

Prof. Pearson's 24 point "mid-panel" and 25 point "mid-point" formulas give the value nearly accurately to the 7th place.

The interesting diagram given on p. 34 of Tract III showing the arrangement of the points of an equi-rectangular network in concentric rings round the central point 0, 0, suggests that two-variable graduation formulas of the summation type might be

constructed in much the same way as linear formulas, the pair of terms $u_{\pm m}$ being replaced by the ring of terms $r_m u_{0,0}$. Thus up to 2nd differences $r_1 = 4 + \delta^2 + \delta'^2$; $r_2 = 4 + 2(\delta^2 + \delta'^2)$; $r_3 = 4 + 4(\delta^2 + \delta'^2)$; $r_4 = 8 + 10(\delta^2 + \delta'^2)$, &c., so that if summation of all terms up to and including the mth ring be denoted by $[m]$, such an operation as $\frac{1}{729}[2]^3[1 + r_1 - r_2]$ would reproduce u correctly up to second differences.

Mr. E. C. Rhodes's Tract on "Smoothing"—No. VI of the series—is devoted mainly to the demonstration and discussion of a new method of graduation suggested by the "least square method of Dr. Sheppard" and by the osculatory method of interpolation. Mr. Rhodes aims at obtaining a graduated series "which is "*reasonably* smooth and which at the same time gives a series of "errors which have a *reasonably* small mean square" by combining the improvement in the individual value produced by Dr. Sheppard's method* with the continuity from value to value given by osculatory (or tangential) interpolatiion. Briefly, he determines u'_{x+1} by a 4th (or other) order parabola passing through the point of which u'_x is the ordinate, having the same tangent at that point as the similar parabola by which u'_x has already been determined and subject to these conditions being the best fit (by least squares) to the $2m+1$ points $u_{x-m} \ldots u_{x+m}$. To start the graduated series of values he determines the first two values—say u'_0, u'_1, if the ungraduated series begins with u_{-m}—by fitting a parabola to $u_{-m} \ldots u_m$ without the specified conditions. The results, so far as may be judged from the test-graduations given in the Tract, seem good, but in the case of the mortality table selected as the principal test the graduated values appear from the following comparison to be hardly so smooth as those obtained, with much less work, by Dr. Sheppard's best 2nd difference method which Mr. Rhodes does not mention but with the results of which those given by the new method (since one of its objects is to obtain a smooth series of values as a whole) may more properly be compared than, as they are in the Tract, with the "best" values obtainable by direct improvement:

* *i.e.*, the method of taking $u' = \Sigma pu$ where Σp^2 is a minimum and differences of order higher than j are negligible, or (what comes to the same thing), of fitting a parabola of order j to $u_{-m} \ldots u_m$ by least squares and taking the central ordinate as u'. For convenience we follow the author of the Tract in referring to this as Dr. Sheppard's method, but it must be remembered that it is only incidentally a method of graduation, being strictly a method of obtaining the best values of the individual u's on certain assumptions as to the nature of their errors, and that if smoothness of the graduated series as a whole—combined with accuracy—is desired Dr. Sheppard recommends—on the same assumptions—the method (to which the summation method at its best is an approximation) of obtaining as accurate values as possible for the differences of u' of order j.

Infantile Mortality (*J.I.A.*, vol. xlviii, p. 184).

Original Values	Graduated Values by			
	New Method $m=6$; $j=4$		Best 2nd Difference Method $m=6$; $j=2$	
	Value	Δ	Value	Δ
124	130·0	−1·6	129·5	−1·3
132	128·4	−1·7	128·2	−1·4
127	126·7	−1·2	126·8	−1·0
130	125·5	− ·8	125·8	− ·4
118	124·7	− ·4	125·4	+ ·1
128	124·3	+ ·2	125·5	+ ·1
125	124·5	+ ·9	125·6	+ ·1
126	125·4	+1·1	125·7	+ ·5
127	126·5	+ ·8	126·2	+ ·6
129	127·3	+ ·8	126·8	+ ·9
127	128·1	+1·1	127·7	+1·3
125	129·2	+1·6	129·0	+1·4
128	130·8	+1·5	130·4	+1·2
135	132·3	+ ·8	131·6	+ ·8
136	133·1	− ·1	132·4	+ ·1
133	133·0	−1·1	132·5	− ·7
131	131·9	−1·8	131·8	−1·6
125	130·1	−2·1	130·2	−1·8
133	128·0	−1·7	128·4	−1·4
127	126·3	−1·2	127·0	−1·0
125	125·1	−1·0	126·0	−1·1
123	124·1	−1·0	124·9	−1·3
123	123·1	−1·2	123·6	−1·4
126	121·9	−1·2	122·2	−1·9
119	120·7	−1·8	120·3	−2·3
118	118·9	−2·8	118·0	−3·0
114	116·1	−3·4	115·0	−3·3
115	112·7	−3·6	111·7	−3·4
107	109·1	...	108·3	...
3,636	3,637·8	...	3,636·5	...
	$\Sigma(u'-u)^2=262{\cdot}2$		$\Sigma(u'-u)^2=259{\cdot}9$	

It is possible that better results might have been obtained by the new method with a lower order parabola. Mr. Rhodes proceeds to investigate the effect of making the successive parabolas osculate instead of merely touch—using in this case a fifth order parabola—but the results are not so good, and he infers that for the data considered in the Tract the limits of good smoothing have been reached by the tangential method and that the extra refinement introduced by imposing contact of the second order does not improve the graduation. It would have been of some interest to know what the effect would have been of imposing one condition less instead of one more, *i.e.*, of making each successive parabola merely pass through the point fixed by its predecessor instead of being tangential.

To test the comparative merits of first or second contact or none in any particular case it would, however, apparently be necessary to determine for each process the most appropriate order of parabola (p. 52).

It is easy to criticise the method proposed by Mr. Rhodes. It is clearly about as unsymmetrical as any method could be. The graduated series reaches out like an extending ladder from the first two values—which will depend as a rule on relatively scanty data—so that (to change the metaphor) the tail seems to some extent to wag the whole dog. Each successive graduated value involves one additional u—the first involving $2m+1$ only and the last the whole series—and by starting at the other end or at any other point a different graduated series would be obtained. Then, although apparently unnecessarily, the parabola determining u'_x is fitted to $u_{x-m-1} \ldots u_{x+m-1}$ instead of to $u_{x-m} \ldots u_{x+m}$. Altogether "there seems"—to use the Editor's criticism of Mr. H. E. Soper's 21-point two-variable interpolation system—"something artificial about the arrangement." But, when all this is said, the facts remain that it gives good, if not specially good, results, and—what is more important sentimentally if not practically—that it satisfies the desire that is felt by some minds for a more direct connection between the successive graduated values than is given by such individualist methods as Dr. Sheppard's or the summation. A connection of this kind could, however, be obtained if it were thought worth while, without entire—or indeed any—sacrifice of symmetry. One plan would be to determine, say, every fifth graduated value independently from the data by Dr. Sheppard's method, and to fill in the intermediate values by tangential or osculatory interpolation combined with parabolic fitting—*i.e.*, to adopt with modifications Mr. King's method—a method to which Mr. Rhodes does not refer but which has something in common with his own. This plan, however, leaves a good deal to the "pivotal" values, and admits of as many different graduated series as there are possible arrangements of pivotal points. A better plan would seem to be to abandon any fixed starting point or fixed pivotal values, and to determine u' by means of a parabola having the unknown graduated values u'_{-1} and u'_1 as ordinates and fitting, subject to this, the $2m+1$ ungraduated values $u_{-m} \ldots u_m$. This leads to a difference-equation of which the solution is

$$u'_x = \mathrm{A}\cos x\theta + \mathrm{B}\sin x\theta$$
$$+ \{\mathrm{K}_0 \sin x\theta + \mathrm{K}_1 \sin (x-1)\theta + \ldots + \mathrm{K}_{x-1}\sin\theta\}/\sin\theta$$

in which $\theta = \cos^{-1}\{1 + \Sigma(r^2-1)^2\}/\Sigma r^2(r^2-1)$,

$$\mathrm{K}_x = 2\Sigma(r^2-1)u_{x+r}/\Sigma r^2(r^2-1)$$

(where Σ denotes summation from $-m$ to m) and A and B are constants of integration which may be used, in the case of mortality data, to make the deviations of the actual from the expected deaths, and the accumulated deviations, zero. The effect is that the graduated

values are given by a series of interlaced parabolas, each successive pair of these parabolas having two points on the graduated curve in common. A short section of the Government Female Annuitants (1883) Ultimate Table has been graduated by Mr. Rhodes's proposed method, the best 2nd difference method and this interlaced parabola method with the following results, to which are added the graduated values by Spencer's 21-term formula (*J.I.A.*, vol. xli, p. 364):

Government Female Annuitants (1883) *Ultimate Table.*
Graduated Values of q_x *for Ages* 50–59.

x	New Method $m=6$; $j=4$		Best 2nd Difference $m=6$; $j=2$		Interlaced Parabolas $m=6$		Spencer's 21-term Formula $m=10$	
	q	Δ	q	Δ	q	Δ	q	Δ
50	1221	127	1247	139	1298	93	1278	104
51	1348	139	1386	156	1391	106	1382	112
52	1487	116	1542	120	1497	106	1494	111
53	1603	89	1662	66	1603	98	1605	102
54	1692	78	1728	71	1701	96	1707	88
55	1770	90	1799	76	1797	90	1795	76
56	1860	112	1875	67	1887	85	1871	69
57	1972	116	1942	79	1972	90	1940	72
58	2088	101	2021	78	2062	98	2012	83
59	2189	...	2099	...	2160	...	2095	...

It seems open to question whether such additional smoothness as may be obtained by Mr. Rhodes's new method or by the interlaced parabola method is worth the labour involved in obtaining it—or in fact whether the sort of smoothness obtained by any method other than fitting is of any value at all. No doubt the true curve is a smooth curve, but it does not follow that because one graduated curve is smoother than another it is a better approximation to the true curve. We can make our results "neater", as Dr. Sheppard has put it in one of his papers, but what is the object of "neatness"? Absolute smoothness, such as is obtained by fitting, has obvious practical advantages. By taking our elementary values to a sufficient number of places we can then rely on values depending on interpolation or differences. But apparent smoothness —mere "neatness"—if it leads to a similar reliance may be a delusion and a snare. There is something to be said for the view that the best graduated values are the best individual values that can be obtained from the data, and that the proper way of neutralising their irregularities is to employ formulas involving a sufficient range of terms. Whether the assumptions that errors are of equal magnitude and that observations are of the same weight form a satisfactory basis for obtaining the best values is another matter.

On the Remainder Form of Certain Formulas of Mechanical Quadrature. By J. F. STEFFENSEN, Ph.D.

[Pp. 9. Reprinted from the Skandinavisk Aktuarietidskrift. Upsala, 1921.]

MOST of the quadrature formulas used in practical work, such as Simpson's, Weddle's and G. F. Hardy's, are obtained directly or indirectly by integrating a substituted parabolic curve determined by an odd number of equidistant values of the function to be integrated*, or by combination of the formulas thus obtained. Dr. Steffensen shows that the remainder-term of formulas of this type can be expressed in the form $Cu^{(n+2)}(\xi)$, where $(n+1)$ is the number of equidistant values involved, C is a numerical coefficient which depends only on this number and can be readily calculated for any given formula by evaluating a simple integral, and $u^{(n+2)}(\xi)$ is the $(n+2)$th differential coefficient of the function for some unknown value ξ of the variable within the range of integration. The analysis by which the result is reached is extremely neat and instructive, and the result itself is interesting on account of its simplicity and generality. But the expression of the remainder-term in this special form does not appear to throw any new light on the relative accuracy of the various formulas. The numerical coefficients C would seem to be necessarily the same as those of the first neglected difference, and the comparative merits of the formulas with reference *inter alia* to the magnitude of the coefficients of the leading neglected differences were fully discussed by Dr. Buchanan many years ago in his paper on "The Use of "Quadrature Formulæ and other Methods of Approximation for "the Calculation of Survivorship Benefits" (*J.I.A.*, vol. xxxvii, p. 384).

Dr. Steffensen infers from his table of coefficients that the formulas of the higher orders are much superior to the formulas obtained by repeated application of Simpson's formula. This may be so as a rule, but the difficulty is to know when we have to deal with an exception to the rule. As Dr. Steffensen states in his next paragraph, "the advantage of using one of the formulas of high "order is lost if the differential coefficients of the function increase "beyond a certain measure." This is strikingly brought out by Dr. Buchanan in the two examples given on p. 388 in the paper mentioned above—the smallest error in one of these examples being shown by a triplicated Simpson. It would seem also that the duplicated Hardy—Mr. King's 39 (*a*)—with 6 terms and a 6th difference error will in many cases give a better result than the straightforward 7-term formula with an 8th difference error. It is doubtful whether it is possible to predict with any certainty the comparative results of different formulas—at any rate without

* The simplest way of doing this seems to be to integrate Lagrange's interpolation formula for u_x in terms of the requisite equidistant values. This method, although referred to by Dr. Steffensen as well-known and although very obvious when suggested, is not mentioned in *Text-Book*, Part II, nor (explicitly) in the comparatively recent Edinburgh tract on Interpolation and Numerical Integration.

preliminary inspection of the differences of the function and its geometrical form.

In the concluding paragraph of his paper Dr. Steffensen draws attention to the need for "caution . . . . in applying quadrature "formulas for calculating benefits by select mortality tables "graduated by G. F. Hardy's method." Dr. Buchanan referred to the point in general terms at the end of his paper in vol. xxxvii, and subsequently (as readers of the *Journal* will have been reminded by Mr. Lidstone's letter, vol. lii, p. 514) the late T. G. Ackland investigated it experimentally in connection with the calculation of joint-life annuity-values and expressed the opinion that "formula 39 (a), . . . is not suitable for the calculation of *select* annuity values." The difficulty, such as it is, can be easily got over by using Simpson for the select period and 39 (a) for the ultimate, but it is doubtful whether this would as a rule be worth while. In theory, and in comparisons of different methods of approximation when relatively small errors may lead to wrong conclusions, any discontinuity in the function is no doubt material, but for practical purposes the slight discontinuity in such a table as the $O^{[M]}$ seems to be of little importance. The $O^{[M]}$ 3 per-cent value of $\bar{a}_{[30]}$, for example, by formula 39 (a) will be found to be only about 3 out in the third place.* The serious difficulties with quadrature formulas appear to arise not so much from slight discontinuities as from the existence of maxima and minima. An unnoticed—or neglected—maximum may cause an error of real importance. Simpson's formula applied to the quadrature of the simple function $x^3(1-x)$ from 0 to 1 gives an error of $16\frac{2}{3}$ per-cent in the result. By applying the formula in sections—

from 0 to the maximum at $x=\frac{3}{4}$ and from the maximum to 1—*i.e.*,

by using the formula $\frac{1}{2}u_{\frac{3}{8}}+\frac{1}{6}u_{\frac{3}{4}}+\frac{1}{6}u_{\frac{7}{8}}$, the error, although not entirely eliminated is reduced to under 2 per-cent. It would appear to be essential, before applying a quadrature formula to a function of unfamiliar geometrical form, to test for maxima and minima.

* The Simpson-39 (a) combination gives the value correctly to the third place.

Compound Interest. *By* A. Skene Smith. *Second and Enlarged Edition.*

[Pp. vi + 63. London: Effingham Wilson, 1922. Price 1*s*. 6*d*. net.]

This is an amusing little book. Among so many things that are good it is difficult to choose, but perhaps the best is an example about the price to be paid by a railway company for 14 acres of land in Marylebone with premises thereon, "the annual rental of which "is £3,610 and some of the leases run for 51 years." One method of calculating the price is to multiply £3,610 by 51, which gives £184,110 "as the capital sum to be paid for the property." (This,

it is stated at the beginning of the chapter, "is a very simple "method, and in practice is found very satisfactory, by increasing "or lessening the number of years"). A second method is to "find "the present value of an annuity of £3,610 for 51 years at 2½ per-cent "per annum." This comes to £103,413. "If, however, the "capitalized sum is to be the *amount* of an annuity, then it will be ". . . £364,334 19*s.* 6*d.* *But,*" the example concludes (the italics are ours) "*the proprietor concluding that the property will increase in "value claims* £400,000." Most railway companies know that proprietor.

There are a few little slips—"principals" (twice) for "principles" in the preface and "the present value of £1" for "the present value "of an annuity of 1 per annum" on p. 63—but these do not detract from the reader's enjoyment.

It is only fair to say that the book has a serious side, but as an attempt to supply "the want of explanation in the ordinary text-books" and to inculcate an elementary working knowledge of compound interest it is not in our view entirely successful. The author's fundamental proposition—the importance of which is indicated exceptionally by the letters Q.E.D. appended to an arithmetical verification in a particular case—that "the amount of "an annuity of any sum at compound interest at a given rate and "time is the present value of an annuity of the amount of the same "sum for the same time and rate" appears to us as likely as anything in the ordinary text-books to lead to cerebral congestion. It would, we think, have been better, instead of showing how examples in compound interest may be worked out by multiplication or logarithms, to give a simple explanation of the nature and use of interest tables. The author states in his preface that "annuity "tables are of great use in practice, but cases often arise in which "they are found deficient." Any deficiencies in interest tables for the simple calculations with which students of this book are likely to be concerned will arise solely from want of practice in using them. It is much more important for practical purposes to teach workers how to use their tools properly than how to make shift without them.

CORRESPONDENCE.

Mr. LIDSTONE'S METHOD OF APPROXIMATING TO THE VALUES OF JOINT-LIFE AND LAST SURVIVOR ANNUITIES.*

To the Editors of the Journal of the Institute of Actuaries.

Dear Sirs,—It has been pointed out to me by Mr. McCormack that some corrections are required in my note in *J.I.A.*, vol. xlvi (p. 168), and I am indebted to him for most of the following errata. Though they do not affect the general argument of the

* *J.I.A.*, vol. xlvi, p. 1.

note I fear that the errors may have caused some trouble to students.

In section 3 it is remarked that "A, B, C, D, can be determined if we know four values of the function not all belonging to the same curve." Instead of *same* curve read *mean* curve. If the values are all on the mean curve, C does not enter into the formula and cannot be determined. It may be necessary to warn students that since the formula is approximate only, the four values should be as near as may be practicable to the value required.

In the same section it is stated that "α, β, γ, lie on the curve obtained by giving $\pm k\sqrt{3+k}$ the value 2, and δ lies on the curve obtained by giving to the same expression the value -2." These statements apply to γ, and to α, β, δ, respectively.

In the numerical example given the value of $10^4\mathrm{D}$ is more accurately $-\cdot 0271$, instead of $-\cdot 0268$.

Section 7 begins, "The expression $12+ak$ is always $+ve$, because ak cannot have a larger value than -2." For -2 read -12, and add the qualification, if $z<6$.

In Section 9 the geometrical interpretation of the results is not correctly given. The intersecting curve which "cuts the lowest curve of the set when $\Sigma a^2=113$; the mean curve when $\Sigma a^2=156$; and the uppermost curve when $\Sigma a^2=216$", is not the parabola representing two terms of Mr. Lidstone's formula, but is the curve representing values which are midway between those given by two terms and by three terms of Mr. Lidstone's formula. When $\Sigma a^2=216$ the parabola does cut the uppermost curve; but when $\Sigma a^2=113$, the parabola and the three-term formula give different values which are equally good approximations to the ordinate of the lowest curve; and when $\Sigma a^2=156$, they give different values which are equally good approximations to the ordinate of the mean curve.

The values of Σa^2 when the parabola cuts the lowest curve, the mean curve, and the uppermost curve respectively are $42^2/_3$, 96, and 216.

I am, Dear Sirs,

Yours faithfully,

D. C. FRASER.

1, *North John Street, Liverpool,*
4 *January* 1922.

ERRATUM.

J.I.A., October 1921, *p.* 488.

The right-hand member of equation (2) should read:

$$\frac{1}{\mathbf{a}_0}\left[1 - c^x\gamma\Delta\mathbf{a}_0/\mathbf{a}_0 - c^{2x}\gamma^2\left(\frac{\Delta^2\mathbf{a}_0}{2\mathbf{a}_0} - \frac{(\Delta\mathbf{a}_0)^2}{\mathbf{a}_0^2}\right)\ldots\right]$$


IN MEMORY OF THE MEN OF THE INSTITUTE OF
ACTUARIES WHO DIED FOR THEIR COUNTRY 1914-1919
JOHN W. E. ALEXANDER
GEORGE F. T. ASCOTT
WILLIAM ASKHAM
THOMAS N. ASKWITH
VINCENT J. AUSTIN
STANLEY O. BENJAMIN
CLAUDE BIDWELL
FRANCIS S. BLAKE
WILFRED BRADLEY
FREDERICK L. BRISTOW
HARRY W. BROWN
GEORGE E. BURROWS
GEORGE L. L. CARTER
ARTHUR V. CLARE
EDWIN C. K. CLARKE
RAYMOND COLE
EDGAR CORBLE
THOMAS G. CUNLIFFE
LESLIE DAVIES
PERCIVAL J. DAVIS
FREDERICK DEFRIES
GERALD D. DOUCET
EDWARD M. DOVE
R. G. GREGSON ELLIS
CHRISTOPHER J. ELLIOT
WALTER S. EMERY
JOHN M. FIELD
GILBERT S. FIELDEN
RICHARD C. FIPPARD
HERBERT D. S. FROMANT
AUSTYN J. C. FYFE
RALPH G. GALE
CHARLES G. GIFKINS
FREDERICK G. GOODYEAR
FREDERICK J. GRANT
GEORGE H. GRANTHAM
HUBERT C. A. GRAVATT
ARTHUR S. GREGORY
MALCOM H. GRIGG
ERNEST H. M. GUMPRECHT
HUGH J. HAMMOND
ALFRED R. HARRIS
LIONEL F. HAWKINS
GEOFFREY Y. HEALD
JAMES HOGG
THOMAS HOLGATE
JOHN C. HURLEY
ALEXANDER JENNINGS
HENRY B. KEABLE
ERNEST C. KEMP
DONALD KERR
J. MILES LANGSTAFF
ROBERT J. LEDGER
MELVILLE E. LOBB
JOHN V. McLEAN
CYRIL P. MADDOX
JAMES H. MARLIN
FREDERICK C. MANN
ERNEST M. MARTIN
THOMAS MIDDLETON
HENRY J. MILLS
BRIAN NEEDELL
EDWARD A. NEWLAND
JAMES H. ORR
HARRY ORRELL
HUBERT H. PHILLIPS
GEORGE H. POLLOCK
GILFRID M. REEVE
DONALD A. ROBERTS
H. T. KAY ROBINSON
CHARLES S. SHILSON
WALTER E. SMITH
SIDNEY F. SNOWDON
ALAN D. STEED
JOHN B. E. TOMBS
JOHN J. TWENTYMAN
HARRY WALLIS
HARRY WATSON
SYDNEY G. WEATHERDON
FREDERICK WELLISCH
EDGAR R. WILLIAMSON
DAVID G. YOUNG

JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

REPORT OF SPEECHES

AT THE

Unveiling of the Institute of Actuaries' War Memorial

AT

STAPLE INN HALL, HOLBORN,

ON

THURSDAY, 16 MARCH 1922, AT 5 P.M.

SIR ALFRED WATSON, President, said: Ladies and Gentlemen, we are met this afternoon in solemn pride to pay our combined tribute to the memory of our gallant brethren who died for England in the Great War, and to unveil the Tablet by which we seek, not to remind ourselves, for we can never forget, but to ensure that those who succeed us in the Institute of Actuaries shall be fully seized of the part the men of the Institute took and of the greatness of the sacrifice so many of them made in a conflict of which the might and the fatefulness will not be diminished in the perspective of history.

War Memorials multiply as, in obedience to a compelling sense of reverence and gratitude, men seek to perpetuate the achievements of those who, forsaking the peaceful tasks of civil life, stepped proudly forward to resist oppression and died in that resistance. In other places the names of most

of those whom we commemorate are doubtless enshrined in loyal and grateful recollection by their associates in work and play. We, nevertheless, must have our Memorial of them. They were a part of our little community, sharing with us the pursuits and interests that bind us together, and by the high courage with which they faced the greater task, and by the sacrifice they made, they have endowed us with a sense of corporate pride to which we must needs give expression. To fight for one's country!—to die in its defence!—what eloquent dissertations, throughout the ages, have been produced upon these themes! But how infinitely different it is from contemplating the supreme sacrifice for country in philosophic abstraction to be brought into contact with it through those who were our intimate associates in our work and in our studies. Not all the ennobling sentiments to which the masters of prose and poetry have given utterance will suffice to express our thoughts and our emotions when we recall, each one of us, some of those whose names are recorded on this Tablet. They died for England! In very truth they died that we might live, that we might hold up our heads as free men, that, undiminished and unsullied, the heritage of liberty which had come to us from our fathers should pass on to our children. Greater even than this—for the fate of England was not the immediate issue—they died to resist the doctrine that might must dominate the world, that power justified aggression, and that, no man hindering, the strong might enslave the weak. Brushing aside all sophistries, and prompt to realize the fundamental issues, masses of our countrymen sprang to arms at the challenge, and none more promptly than the men of the Institute of Actuaries. What they undertook, what they endured, can in no way be so impressively indicated as by the long list of those to whose memory we now do honour.

You are aware that some 430 of our colleagues joined the Forces. Our Memorial Tablet contains 82 names, a high proportion indeed. The cruel toll of high-spirited youth that war remorselessly takes is significantly indicated by the fact that while six of the fallen were Fellows and 16 were Associates, 22 of our Students and 38 Probationers lost their lives. Not all were British in the narrower sense of the word. It is an additional bond between us and our professional brethren over seas that our Tablet includes the names of two

Canadians and five Australians. Each arm of the Forces is represented in the list. While, inevitably, the Army preponderates with 76 names, the Navy claims four and the Air Force two.

Our pride in their gallant service would be increased, if that were possible, by the record of the decorations conferred upon some of them. H. T. K. Robinson, for services described in the Presidential Address of my predecessor, was awarded the Distinguished Service Order, with two bars. The Military Cross was conferred in four cases: E. C. Kaye Clarke, E. M. Dove, F. G. Goodyear and E. R. Williamson. J. J. Twentyman received the Distinguished Conduct Medal. This may not represent the complete list. If so we should be glad to be notified of any additions which should be made to it.

Their resting places are widely scattered, and from what we know of the service of those who, happily, have returned to us it is probable that wherever Englishmen fought some of our men laid down their lives. Certain contests of imperishable memory will have a special significance for us of the Institute. Some of those who were our colleagues in the peaceful life at home lie in the hallowed acres that thickly stud the fields of Flanders, that land of proud and sacred memories for every man of British blood. Ten fell in the first battle of the Somme, one went down with his ship in the battle of Jutland, some gave up their lives in the Dardenelles and on the Gallipoli Peninsula, while others were killed in action or died from the hardships of the campaign based on Salonica.

As your President I have endeavoured in these few words to express, however inadequately—and on such a subject adequate utterance I can never find—the sense of pride that animates us as an Institute in reflecting upon the valour and selflessness of those whose names will have an honoured place on our walls so long as the Institute exists. That my poor words should not seem all there is to say on this historic occasion I am going to ask Mr. Higham, as representing our senior members to address you, and following him Mr. Clement Hall, selected by the Students' Society to speak for the juniors, so many of whom, including himself, did gallant service for their country in its hour of need.

Mr. C. D. HIGHAM: It is strange to find myself talking

again in this Hall, for I thought I had sung my swan song years ago. I do not forget how often "superfluous lags the veteran on the stage," but when you, Sir, with kindly courtesy, suggested it was desirable that one of the older generation should have a share in the proceedings this afternoon I could not but comply, and I think I fairly may, because there are only two Fellows senior to myself. I doubt if some of you younger men knew what it was, when our dear England made her call in 1914, for us who were older to be able to do so little—some financial help perhaps, some entertaining, driving out the wounded, listening to interminable talk on Committees, and so on. But our limitations only increased our pride and pleasure when we saw the splendour of the unrivalled response that you made when you gave everything—your all. As you said, Sir, I do not think any profession excelled ours in the willingness with which the men came forward. I have no reason to suppose that my own office was different from any other, but except those who from age could not go everyone was willing, and everyone served except the few that we had to retain to keep the doors open. And they went out of their way to go. A dear friend of mine in the office, who was refused through an ailment, got admitted into hospital for an operation so as to qualify. He went out, and alas his name is on that list. But when we talk of the heroism of those who went I do think we also want to remember the heroism of those who stayed—I mean the wives who never kept their husbands back, the wives who took up unaccustomed burdens and manfully—that is the word—carried on. "They also serve who only stand and wait", and to my mind the bravery of patience is not less to be commended than the bravery that amid the rush of events and the joy of action goes forward to victory. There is a quotation I have made in this Hall before and I might venture to make it again. You will remember that when Horace was considering what should be his own monument he said "Non omnis moriar, multaque pars mei vitabit Libitinam." You have the same thought in St. Augustine's Confessions: "Nec omnino moriebatur." You have those first three words on Haydn's tomb and in other places, and I often think of them, especially on occasions like this. These men we are commemorating are not dead but living, living in the hearts of those they love—not loved—and who love

them, living in the memory of the companions with whom they worked in this great city and elsewhere, living in this Hall where they began their studies and went on ever learning day by day. And they live for an inspiration and encouragement to us and to all those who after us shall foregather in this Hall. It is not probable that during my lifetime we shall hear the call again in exactly the same way, but duty has many voices, and I have no doubt that when its clarion does ring out, at whatever time and in whatever manner, this profession of ours of which we are so proud will not fail to respond to its great traditions and once again stand forth as a pattern of unselfishness, of patriotism, and of honour. Once more, Sir, I thank you for letting me have the opportunity of adding this pitiful tribute to the memory of the friends we shall never see here again.

Mr. CLEMENT HALL: Much as I appreciate the honour of speaking on behalf of the younger Members of the Institute on this occasion I would that the task had fallen to someone more fitted to do justice to it. It was a privilege to many of us to count as friends those whom we are here to honour to-night, so that in giving expression to our feelings of admiration for them and for what they did we can do so with the certainty that comes from personal knowledge. Of necessity the Memorial makes no reference to the individual deeds of valour performed by those whom it commemorates. Those men, I am sure, would have had it so. When the call came to them they went forth, with no idea of personal gain or glory, to do whatever their country in her need might require of them. To them it was sufficient to know that they were doing their duty. That done they desired no praise. The decorations and medals which had been won by them are a measure of the courage of those who have passed away, but it must not be supposed to represent the sum total of the gallant deeds which have been performed. Many a man who went to his grave with no ribbon upon his breast was in valour the equal of or superior to those whose courage had been officially recognized. I say that with no desire to reduce in the eyes of anyone the value of those decorations which have been granted, but rather with the view to assist you to realize the wonderful quality of those to whom this Memorial is raised. Although this is an occasion upon which our thoughts naturally tend to dwell upon the past we shall, I think, best

keep green the memories of our friends by upholding in the eyes of the actuaries of the world the prestige of this Institute, and by a determination that the ideas of liberty and freedom for which they fought and died shall never be forgotten.

The SECRETARY read the list of names of those inscribed upon the Tablet.

The PRESIDENT, in unveiling the Tablet, said: It is now my duty to unveil this Memorial and hand it to the Institute as a sacred trust, to be held, for ourselves in remembrance, and for future generations of actuaries to show the quality, the patriotism and the strength of purpose, of those men of the Institute who died for their country in the Great War.

I desire to acknowledge on behalf of the Institute the skilled art and craftmanship to which we owe this beautiful Tablet. To Mr. Paul Waterhouse, the President of the Royal Institute of British Architects, whose son Mr. Michael Waterhouse is with us this afternoon, for the design and supervision of the work, and to Messrs. Hart, Son, Peard & Co., for the construction of the Tablet, I tender the sincere thanks of the Institute.

On a Short Method of Constructing Select Mortality Tables. Further Developments. By GEORGE KING, F.I.A., F.F.A., F.A.S., *Consulting Actuary.*

1. A YEAR ago I had the honour of submitting to the Institute a paper on "A Short Method of Constructing Select Mortality Tables", which has been published in the *Journal*, vol. lii, pp. 286–341. For convenience it may be referred to as "the former paper." A definition of the short method was given as follows:

> The short method of constructing select tables consists in preparing first of all an ultimate table, and then in constructing the select columns by means of factors formed from the ratios between the actual deaths and those expected by the ultimate table, and by which factors the ultimate q_x is to be multiplied.

In the definition nothing is said as to how the factors should be formed from the ratios, or as to the form which a column of factors should take.

2. The illustrations in the former paper were limited to tables constructed by means of factors uniform for each column of the select period, the total actual deaths of a particular column being divided by the total ultimate expected deaths, and the resulting ratio being used as a factor constant for the whole column. The graduated rates of mortality age by age were therefore a uniform percentage of the ultimate rates of mortality. The Experiences used for the principal illustrations were the British Offices' Experience and the Experience of the Australian Mutual Provident Society; and in both cases the results were found to be surprisingly good, so much so that select tables constructed in this very simple way might probably be used in many cases without much misgiving for all ordinary purposes. Moreover, the remarkable fact was brought out that for the first five years of the select period of the British Offices' Experience, the new table was almost identical with the official table.

3. It was not intended, however, that this should be the final stage, and at the outset in the former paper it was said:

> No matter what may be the entry age, all the lives passing through the select period are acted upon by very similar forces; and, presumably, the rates of mortality are thereby affected in similar ways, and in the same directions, but it must not be assumed without enquiry that they are affected in exactly the same proportions, as is done when a uniform factor is employed for the whole column. It may be that the factors should increase, or should decrease, or should vary otherwise, with advancing entry age.

4. Later on in the former paper it was also said of the British Offices' Experience that allowance could be made "without difficulty" for the tendency in some of the assurance years towards increase in the factors with advancing entry age, by taking the three ratios for the age groups 20 to 34, 35 to 49, and 50 to 64, and transforming them into factors for individual ages. The view that there would be no difficulty had, however, been found, by the time the paper was submitted to the Institute, to have been too sanguine; and in replying to the discussion I was obliged to confess that there was great difficulty, because a double graduation was involved, perpendicular and horizontal; and the introduction of a variable factor, or any changes in the perpendicular graduation, affected the horizontal graduation.

5. The reception of the former paper was very cordial, and I cannot sufficiently thank my professional brethren for their

kindness to me. It has led me to continue the search for a solution, and I have now the honour to submit further developments of the short method of constructing select mortality tables, by which it is hoped that the difficulties of the double graduation have been to a great extent overcome.

6. Table I appended gives in age groups (the first group including three ages, and the others five ages each) for each of five select columns, and for the total of each column, the ratios of the actual deaths to the ultimate expected deaths for both the $O^{[M]}$ Experience and the $(AMP)^{[M]}$.

7. In the case of the $O^{[M]}$ Experience, with a range of 48 ages, the ratios of the totals of the columns progress from column to column with sufficient regularity, and do not require adjustment. They are as follows:

Year	Ratio	Rate of Progression
1st	·44299	1·4738
2nd	·65287	1·1361
3rd	·74169	1·0628
4th	·78824	1·0314
5th	·81297	...

8. In the case of the $(AMP)^{[M]}$ Experience, however, it is different, and in the former paper an adjustment was made by transferring 14 deaths from the 3rd year of selection to the 4th; but the rate of progression is improved a little by the further adjustment of transferring 2 deaths from the 4th year to the 5th. The following are the figures for a range of 48 ages:

Year	Unadjusted		1st Adjustment		2nd Adjustment	
	Ratio	Rate of Progression	Ratio	Rate of Progression	Ratio	Rate of Progression
1st	·63804	1·2149	·63804	1·2149	·63804	1·2149
2nd	·77516	1·1146	·77516	1·0753	·77516	1·0753
3rd	·86398	·9650	·83351	1·0382	·83351	1·0328
4th	·83371	1·0512	·86543	1·0128	·86082	1·0236
5th	·87641	...	·87641	...	·88111	...

9. A range of 48 ages is a little too long for the $(AMP)^{[M]}$ Experience, and better results are derived from a range of 44 ages. The following are the figures for this shorter range. They include the second adjustment of par. 8, a transfer of

14 deaths having been made from the 3rd column to the 4th, and of 2 deaths from the 4th column to the 5th :

Year	Unadjusted		Adjusted	
	Ratio	Rate of Progression	Ratio	Rate of Progression
1st	·63694	1·2100	·63694	1·2100
2nd	·77069	1·1105	·77069	1·0705
3rd	·85589	·9714	·82508	1·0409
4th	·83143	1·0518	·85884	1·0238
5th	·87452	...	·87928	...

10. It is assumed that the ratios increase year by year as time passes, from the moment when the lives first come under observation, until the effects of selection are exhausted. The select columns will then merge into the ultimate table, and the ratios will reach their limiting value of unity.

11. There may, however, be cases following a different law. For instance, it has been shown by Mr. Kenchington, *J.I.A.*, vol. xliv, Table 6, p. 150, that, when female lives are in question, the rates of mortality at certain ages may fall with the lapse of time owing to a diminishing maternity risk ; and under such conditions the ratios would decrease for a few years. That case, however, is special, and would have to be specially treated. In every case the first step must be to examine the progression of the ratios, and to make such adjustments as may be found to be required.

12. If a column in the select period be divided into sections of t ages each, and if for each section the number of the actual deaths be divided by the number of ultimate expected deaths, we shall obtain ratios which may be written R_1, R_2, R_3, and R_4. Four sections are assumed for the moment because, on account of irregularities in the data, it will seldom, if ever, be feasible to use more. If now in each section the ultimate q at each age be multiplied by the R of the section, we shall have the select q, which may be written q'. In fact, in the ratios, R, we have the factors, F, uniform for each section of the column, by which to multiply the ultimate q in order to obtain the select q' ; and if at each age the exposed to risk be multiplied by the corresponding q' so formed, the actual deaths will be, in the aggregate of each section, accurately reproduced.

13. A table constructed in this way would not, however, be satisfactory, because there would be breaks in continuity at the points of junction of each pair of adjacent sections. The

problem therefore is: From the values of R to construct values of F, which will run smoothly without break from F_1 to F_{4t}, and which will give such values of q' as will in the aggregate of each section reproduce the actual deaths. The ratio, R, being the uniform factor for the t ages of a section, the desired result will be attained with close approximation if we so arrange that

$$tR_1=\Sigma_1^t F,\ tR_2=\Sigma_{t+1}^{2t} F,\ tR_3=\Sigma_{2t+1}^{3t} F, \text{ and } tR_4=\Sigma_{3t+1}^{4t} F.$$

This can be done by expressing the values of F_1, the initial F, and its leading differences, in terms of the given values of R and their differences. The differences of R will be symbolized by Δ, and those of F by δ.

14. By the ordinary summation formula of Finite Differences (*Text-Book*, Part II, Chap. xxiv, formula 13)

$$tR_1=tF_1+\frac{t(t-1)}{2}\delta+\frac{t(t-1)(t-2)}{6}\delta^2+\frac{t(t-1)(t-2)(t-3)}{24}\delta^3$$

Here the coefficient, t, is common to both sides of the equation, and cancels out, and we have

$$R_1=F_1+\frac{t-1}{2}\delta+\frac{(t-1)(t-2)}{6}\delta^2+\frac{(t-1)(t-2)(t-3)}{24}\delta^3$$

Similarly, the coefficient, t, disappears when R_2, R_3, and R_4 are involved.

15. We therefore have the following scheme:

$$R_1=F_1+\frac{(t-1)}{2}\delta+\frac{(t-1)(t-2)}{6}\delta^2+\frac{(t-1)(t-2)(t-3)}{24}\delta^3$$

$$R_1+R_2=2F_1+\frac{2(2t-1)}{2}\delta+\frac{2(2t-1)(2t-2)}{6}\delta^2+\frac{2(2t-1)(2t-2)(2t-3)}{24}\delta^3$$

$$R_1+R_2+R_3=3F_1+\frac{3(3t-1)}{2}\delta+\frac{3(3t-1)(3t-2)}{6}\delta^2+\frac{3(3t-1)(3t-2)(3t-3)}{24}\delta^3$$

$$R_1+R_2+R_3+R_4=4F_1+\frac{4(4t-1)}{2}\delta+\frac{4(4t-1)(4t-2)}{6}\delta^2+\frac{4(4t-1)(4t-2)(4t-3)}{24}\delta^3$$

16. Multiplying out the numerators of the expressions on the right of the equations, and differencing four times, we have :

$$R_1 = F_1 + \frac{t-1}{2}\delta + \frac{t^2-3t+2}{6}\delta^2 + \frac{t^3-6t^2+11t-6}{24}\delta^3$$

$$R_2 = F_1 + \frac{3t-1}{2}\delta + \frac{7t^2-9t+2}{6}\delta^2 + \frac{15t^3-42t^2+33t-6}{24}\delta^3$$

$$R_3 = F_1 + \frac{5t-1}{2}\delta + \frac{19t^2-15t+2}{6}\delta^2 + \frac{65t^3-114t^2+55t-6}{24}\delta^3$$

$$R_4 = F_1 + \frac{7t-1}{2}\delta + \frac{37t^2-21t+2}{6}\delta^2 + \frac{175t^3-222t^2+77t-6}{24}\delta^3$$

$$\Delta R_1 = t\delta + t(t-1)\delta^2 + \frac{14t^3-36t^2+22t}{24}\delta^3$$

$$\Delta R_2 = t\delta + t(2t-1)\delta^2 + \frac{50t^3-72t^2+22t}{24}\delta^3$$

$$\Delta R_3 = t\delta + t(3t-1)\delta^2 + \frac{110t^3-108t^2+22t}{24}\delta^3$$

$$\Delta^2 R_1 = t^2\delta^2 + \frac{36t^3-36t^2}{24}\delta^3$$

$$\Delta^2 R_2 = t^2\delta^2 + \frac{60t^3-36t^2}{24}\delta^3$$

$$\Delta^3 R_1 = t^3\delta^3$$

17. Whence, by transposing the terms of the equations,

$$\delta^3 = \frac{\Delta^3}{t^3}$$

$$\delta^2 = \frac{\Delta^2}{t^2} - \frac{3(t-1)}{2}\delta^3$$

$$\delta = \frac{\Delta}{t} - (t-1)\delta^2 - \frac{(t-1)(7t-11)}{12}\delta^3$$

$$F_1 = R_1 - \frac{t-1}{2}\delta - \frac{(t-1)(t-2)}{6}\delta^2 - \frac{(t-1)(t-2)(t-3)}{24}\delta^3$$

18. The calculation of F_1 and its differences, although simple in theory, is a little tricky in practice, and mistakes are easily made ; and the following equation, given in very slightly different form in par. 15 above, can be used conveniently for checking the results before actually constructing the column of F :

$$R_1+R_2=2F_1+(2t-1)\delta+\frac{(2t-1)(2t-2)}{3}\delta^2+\frac{(2t-1)(2t-2)(2t-3)}{12}\delta^3$$

19. So far it has been assumed that the column has been divided into four sections, but the investigation applies equally if there be three sections, or two sections. When there are three sections the terms involving Δ^3 and δ^3 vanish, and when there are two, the terms involving Δ^2 and δ^2 also vanish; but otherwise the formulas in their algebraical form remain unchanged.

20. For all the experiments on the British Offices' Experience a range of 48 ages was used, and it may be a convenience to others to have a record of the numerical coefficients of the formulas in pars. 17 and 18. They are as follows:

(*a*) Four sections of 12 ages each.

$$\delta^3=\frac{\Delta^3}{1728};\quad \delta^2=\frac{\Delta^2}{144}-16{\cdot}5\delta^3;\quad \delta=\frac{\Delta}{12}-11\delta^2-66{\cdot}91\dot{6}\delta^3$$

$$F_1=R_1-5{\cdot}5\delta-18{\cdot}3\dot{3}\delta^2-41{\cdot}25\delta^3$$

$$R_1+R_2=2F_1+23\delta+168{\cdot}6\dot{6}\delta^2+885{\cdot}5\delta^3$$

(*b*) Three sections of 16 ages each.

$$\delta^2=\frac{\Delta^2}{256};\quad \delta=\frac{\Delta}{16}-15\delta^2;\quad F_1=R_1-7{\cdot}5\delta-35\delta^2$$

$$R_1+R_2+R_3=3F_1+70{\cdot}5\delta+1081\delta^2$$

(*c*) Two sections of 24 ages each.

$$\delta=\frac{\Delta}{24};\quad F_1=R_1-11{\cdot}5\delta;\quad R_1+R_2=2F_1+47\delta$$

(*d*) If there be only one section, we have the uniform factor employed in the former paper.

21. In applying the formulas in practice it is necessary to work to seven places of decimals, especially when there are four sections to the columns, in order to get the values of F correct to five places. This is obvious when it is remembered that, with a range of 48 years in a column of four sections, the last value of F involves δ^3 no less than 16215 times. When,

however, the work of constructing the table by the addition or subtraction of the differences is performed on the arithmometer, the extra places of decimals give very little trouble.

22. In order to check the work as it proceeds, it is well to calculate, and to enter on the working sheet, isolated values of F at convenient intervals. This is done by the usual formula

$$F_{1+n}=F_1+n\delta+\frac{n(n-1)}{2}\delta^2+\frac{n(n-1)(n-2)}{6}\delta^3$$

Also, the fundamental relationship of par. 13, that the sum of the F's in each section is equal to t times the corresponding R, affords a complete check for each section as it is finished, although there may be a small discrepancy in the equation for the 4th section, even when seven decimal places have been retained in the work, the numerical coefficient of δ^3 in that section being 143,681.

23. Table I shows that when small age groups are dealt with, the ratios run very irregularly, both perpendicularly down the several columns, and horizontally from column to column. These irregularities are reduced, but do not disappear, when larger groups are taken. This is shown in Table II, where columns of 48 ages each of the $O^{[M]}$ Experience are divided into four sections of 12 ages each. In the first part of the table the columns are treated perpendicularly, and in the second part, horizontally. The 1st year after entry is excluded, as that year can often be treated apart, although sometimes it may be brought in with advantage. The figures, therefore, relate to the remaining four years of the select period. The ratios are those of the actual deaths to the deaths expected by the ultimate table, and in the first part of the table the " proportion " is that which the ratio for the particular section bears to the ratio R_x for the whole column; while, in the second part, the " rate of progression " is simply the ratio on one line divided by the ratio on the line next above it. The AMP Experience shows similar, but even more marked, irregularities.

24. The perpendicular irregularities can be removed from each column separately by the graduation formulas of par. 17; but that leaves the several columns in, as it were, watertight compartments, and the horizontal irregularities are not eliminated. The result is that, when a complete table is constructed on these lines, the columns are not co-ordinated, and do not harmonize with each other, and many serious

anomalies remain, and the table is unsuitable for practical use. Graduation must be effected horizontally as well as perpendicularly, and a moment's thought shows that horizontal graduation is legitimate. With lapse of time the observed lives do not pass down the select table as they do in the case of an aggregate table ; but they pass across the table, and abnormal mortality at one part of a column does not affect the lower part of that column, but it affects the next succeeding columns, and it can be corrected only by lateral graduation.

25. When, in the second part of Table II, the rates of progression of the R's are examined, there is no law revealed. For instance, looking at the 1st and 2nd sections, it is seen that in the 1st section the rates of progression first decrease and then increase, whereas in the 2nd section they constantly decrease. Again, looking at the 3rd and 4th sections, it is seen that in the 3rd the rates of progression first increase, and then decrease, whereas in the 4th they first decrease, and then increase. It seems clear that these anomalies can be due only to defects in the data, and that they cannot be due to any underlying principle determining the sequence of the successive columns. Therefore they should be removed. No law is revealed when the columns are viewed separately, but a law becomes apparent when the select portion of the Experience is taken as a whole ; and that law may be applied with propriety to each column individually. The total mortality of each column will not be changed thereby, because that is fixed in the manner discussed in pars. 8 and 9 ; but it will be redistributed to conform to the law derived from the whole of the data, and uniform for all the columns.

26. A detailed example of this process, which is called "Grading", may be found useful, as thereby the process can be followed without difficulty. This is given in Table III. It relates to the four columns for the 2nd to the 5th years of the $O^{[M]}$ Experience for the 48 entry ages 17 to 64, with the columns divided into four sections, each including 12 entry ages ; and the four columns for the years 2 to 5 of the select period are graded. In each section the actual deaths are brought together from all the four columns, as also the ultimate expected deaths ; and the ratios of the total actual to expected deaths are taken, giving ${}_{\alpha}R_1$ to ${}_{\alpha}R_4$. At the foot of the table the ratio ${}_{\alpha}R_{\infty}$ is given, that is, the ratio for the whole experience including all the sections, and all the four columns. Next, each sectional ratio is divided by the total ratio to give the proportions, written

r_1, r_2, &c., of the sectional ratios to the total ratio. Lastly, in Table IV, the total ratio for each column is multiplied by these r's to obtain the graded sectional ratios, which, by graduation, will be converted into factors.

27. The results of the grading are given in Table IV, and it will be seen how accurately the actual deaths for the totals of the columns are reproduced. In the sections the graded deaths differ somewhat from the actual deaths. At one part of a column a few deaths are transferred to the next column, and at another part the debt is repaid; and in this way the horizontal irregularities of the data are corrected.

28. When a table has been graded in this way, the proportions of the ratios of the successive sections, passing down the table, to the ratios for the totals of the columns are the same for all the columns, and the same for the entire table when the columns have been combined. Also, passing across the table, the rates of progression of the ratios of each of the several sections from column to column are the same for all the sections, and also for the totals of the columns. The words "are the same" have been used, although small irregularities may appear, owing to the fact that the ultimate expected deaths have been taken to only one place of decimals. At the foot of Table II the results of the grading are shown, and these should be compared with the ungraded figures above them.

29. The name "grading" has been given to the process, because the respective gradients of the several curves representing the successive columns of factors bear, at all points, a constant relationship to each other.

30. When the columns of a table have to undergo a preliminary adjustment, grading greatly facilitates the process. For instance, in the case of the (AMP)[M] Experience, fourteen deaths are transferred from the column for the 3rd year to that for the 4th, and when the columns are treated separately it is very difficult, if not impossible, to distribute correctly the transferred deaths among the different sections; but, when the columns are graded, the difficulty disappears, because the distribution is effected automatically.

31. For the O[M] Experience columns of 48 ages were taken, entry ages 17 to 64, because, with 48 ages, we can have two or three or four sections, and it was desired to make various experiments based upon exactly the same material, so that they should all be comparable each with the others. The next

greater or less ranges of table possessing this advantage would include 60 ages or 36 ages, and the experience is not sufficient to warrant the use of so many as 60, while a range of only 36 would be too short for practical purposes.

32. It is not necessary that, for all the columns of a table, the same number of sections should be used; and for the $O^{[M]}$ Experience it was found that for the first year after entry two sections give probably as good a graduation as is attainable, whereas, for the other four years graded together, four sections yield the most satisfactory results. For both the experiences, $O^{[M]}$ and $(AMP)^{[M]}$, for all the years after the first, four sections seem to be better on the whole than any smaller number. Therefore for the new construction of the British Offices' Select Tables the first column was graduated independently, two sections being used, and the other four columns were graded, and four sections were used. In Tables V, (A) to (E), full details are given of all the five columns, and in Table VI, a summary of the whole.

33. The factors of the first column were calculated by the graduation formula of par. 20, applicable to a column of two sections, and ${}_1F_{17}=\cdot3865668$ and $\delta=+\cdot0028375$. The work was done on the arithmometer to the full seven places of decimals, but the results are recorded only to five places. The graduated expected deaths are obtained by multiplying the factor into the $O^{M(5)}$ expected deaths, and the graduation is thus tested before calculating q'.

34. The factors for the 2nd column, the first graded year, were calculated in the same way, only that the formula involving three differences was employed, and that eight decimal places were used on the machine. This gives:

$$\delta^3=-\cdot00001366 \qquad \delta^2=+\cdot00026458$$
$$\delta=-\cdot00123647 \qquad {}_2F_1=+\cdot64315334$$

35. For the other three graded years a much shorter process is available, there being no need to employ for them the graduation formula. The column of F in the second graded year is formed by multiplying the F's for entry age in the first graded year by a constant, the relationship between the columns being:

$${}_3F={}_2F\times({}_3R_\infty\div{}_2R_\infty)$$

Similarly

$${}_4F={}_2F\times({}_4R_\infty\div{}_2R_\infty),\text{ and }{}_5F={}_2F\times({}_5R_\infty\div{}_2R_\infty)$$

The actual multipliers for the $O^{[M]}$ Table are

$$_3F = {_2F} \times (\cdot 74169 \div \cdot 65287) = {_2F} \times 1\cdot 136050$$

$$_4F = {_2F} \times (\cdot 78824 \div \cdot 65287) = {_2F} \times 1\cdot 207346$$

$$_5F = {_2F} \times (\cdot 81297 \div \cdot 65287) = {_2F} \times 1\cdot 245225$$

36. As to the mechanical arrangements, on the working sheets a space separates the figures for the different sections in the columns, and the column of F for the first graded year, with its 1st differences, is written on a movable slip spaced so as to fit into the working sheets. The slip is placed in position on the sheet for the second graded year, and so on in turn for the other graded years. The constant multiplier is placed on the fixed plate of the arithmometer, and the first value is formed, and left on the slide of the machine, and the succeeding values of F are formed by the continued multiplication of the differences on the slip. For checks on the work as it proceeds reference is made to par. 22 above.

37. It must be carefully noted that to obtain the F's of the succeeding graded columns the F's for *entry* age of the first graded column must be multiplied by the constant, and that, to obtain the graduated q', the q of the ultimate table at the *attained* age must be multiplied by the F.

38. Of course the new process of combined graduation and grading is longer than when a uniform multiplier for each column is used. Factors have to be prepared by means of which the graduated q' has to be calculated. Nevertheless it is not intricate, and is not in itself lengthy; and a complete table of five columns can be constructed in quite a limited number of hours. Therefore the original title of "A Short Method of Constructing Select Mortality Tables" does not become inappropriate.

39. In the former paper it was shown, when a uniform factor for each column was used, that the mortality for entry ages 45 to 64 was persistently understated, and that the official tables constructed by the late Sir G. F. Hardy had the same defect, and almost exactly to the same extent. This, as quoted below, is brought out clearly in Table VII (F) of the former paper, *J.I.A.*, vol. lii, p. 320. Table VI of the present paper shows that that defect has now been remedied. The following are the figures. They give, for the whole five columns, the amount of deviation of the graduated deaths from the actual deaths.

Entry Ages	Deviation		
	Official Table	Uniform Multiplier	New Grading
45 to 49	−38·9	−38·8	− 0·6
50 ,, 54	−87·3	−87·1	−50·3
55 ,, 59	−17·8	−17·6	+ 8·0
60 ,, 64	−13·4	−13·7	− 2·6

40. In the new grading there is still an apparent deficiency of 50·3 deaths for entry ages 50 to 54, but, in pars. 45 and 46, lower down, that is explained, and shown to be a necessary correction for irregularities in the original data.

41. In the new construction of the $O^{[M]}$ Table, Table V, the factors for the first year increase uniformly throughout by a small constant difference. In the graded columns for the 2nd to the 5th years the progression varies. For all these columns, the factors diminish at a decreasing rate from entry age 17 to entry age 23, where there is a minimum point, when they begin to increase, at first slowly, and then more rapidly, until entry age 37, when the rate of increase slackens until at entry age 52, where there is a maximum point, the factors begin actually to decrease, and at an accelerating rate to the end.

42. Many experiments were tried with both the $O^{[M]}$ Experience and the $(AMP)^{[M]}$. First, it was attempted to form a complete table of five columns by graduating each column separately. When the columns were examined individually, the results had the appearance of being good; but when they were placed side by side, the table as a whole was found to be useless for practical purposes, because the columns ran counter to each other. Therefore, all that work had to be scrapped. Attention was then directed towards horizontal graduation, but without success, until the method of grading was devised. Then that method of grading was applied in various ways, but some of the resulting tables were defective, and they too were scrapped. Seven for the $O^{[M]}$ Experience were, however, more or less successful; and, for all the seven, complete tables were constructed. No labour was spared, as it was thought well to carry out these exploratory investigations thoroughly once for all. It was hoped that thereby much trouble would be saved to other workers. A useful implement would be placed in the hands of actuaries, who would be able to employ it with confidence, and with some certainty of success, and with a

minimum of labour. They would have merely to study the particular case before them by means of such a table as No. 1 appended, so as to ascertain the range of table to be used, and which of the grading combinations would best suit it, and then to construct the table without much preliminary testing. For a small experience probably a shorter range than 48 ages might be taken, and two sections for the columns might possibly be appropriate, while for a larger experience, three sections, or even four, might be used with advantage. Also, it would not be very troublesome to make an examination to judge whether only four columns should be graded, the first column being treated separately, or whether it would be better to grade all the five.

43. Possibly in the case of a table designed for practical use, even should the Experience be large, it may be found that a very good plan is, boldly to grade all the five columns. It is true that by doing so the result in the first column is not quite so good as when that column is graduated separately, but for practical purposes that does not matter much. No one would think of using the first column of a select table based on whole-life policies to obtain the rate of premium for a one year's term assurance; and, in fact, select tables are not very suitable for such policies. Select tables are intended for whole life cases, and possibly also for endowment assurances, although the endowment assurance mortality is not necessarily the same as the whole life mortality; but any error involved in using a whole-life select table would be on the safe side. When all the five columns are graded together, we shall have the values accurately, at the moment of selection, of whole-life premiums of all kinds, and whole-life annuities, and that is really what is required. For valuation purposes also, the table graded for five years would be quite suitable, and even for a whole-life policy by limited payments it could be used with absolute confidence. That being so, I do not see any objection to grading the whole five years, and by following that course a certain amount of trouble is avoided. The comparative results of grading all the five columns, or only four, are shown in Table VII.

44. As regards the $O^{[M]}$ Experience, seven different graduations were completed, as mentioned above, and it would I think be useful and instructive to attach a summary of each of these to this paper in the form of Table VI appended, which relates

to the new construction of the O[M] Table adopted as apparently the best. I am afraid, however, that the Editors of the *Journal* would object to so much tabular matter being offered to them, and that we must content ourselves with the briefest possible summary of the general results, and that is given in the annexed Table VII. It shows, for each of the seven graduations, and also for a graduation by a constant factor for each column, the total deviation of the graduated expected deaths from the actual deaths for all the five years of the select period taken together, as in the last column for all the five years of the select period in Table VI. The graduations illustrated are as follows:

(1) Uniform factor for each column.
(2) Two sections of 24 ages each, four years graded, the first column being treated separately.
(3) Two sections of 24 ages each, five years graded.
(4) Three sections of 16 ages each, four years graded, the first column being treated separately.
(5) Three sections of 16 ages each, five years graded.
(6) Four sections of 12 ages each, four years graded, the first column being treated separately.
(7) Four sections of 12 ages each, five years graded.
(8) The special table adopted as the best, the first column being graduated separately, with two sections of 24 ages each, and the other columns being graded together with four sections of 12 ages each.

45. It will be seen how consistent in the main are the results of all the eight graduations, the deviations having for the most part the same sign in all of them, and not differing very much in magnitude, although, in so far as the O[M] Experience is concerned, of course some of the methods of graduation are much better than others. This suggests that the deviations are really due to roughness in the original data, and the fact can be proved by forming the ratios of graduated deaths to the ultimate expected deaths in respect of each group of entry ages for the whole of the five years of the select period. The following are the figures, (*a*) for the original data; (*b*) for the full construction given in appended Table VI; and, (*c*) for construction No. 3 by two sections graded all the five years. These two specimens, (*b*) and (*c*), of the graduated tables are used to illustrate all the eight, because all of them have very much in common.

Entry Ages	Average Ratio for the Five Years of the Select Period		
	(*a*)	(*b*)	(*c*)
17 to 19	·72133	·65368	·64785
20 ,, 24	·69082	·65510	·65482
25 ,, 29	·64018	·66332	·66500
30 ,, 34	·67465	·67316	·67431
35 ,, 39	·68315	·68448	·68362
40 ,, 44	·67896	·69575	·69302
45 ,, 49	·70563	·70535	·70188
50 ,, 54	·74198	·71131	·71052
55 ,, 59	·70650	·71377	·71967
60 ,, 64	·71196	·70797	·72731

46. In Table VII all the graduations have large negative deviations for entry ages 17 to 19, and 20 to 24, and in looking at the above ratios it will be seen that in (*a*), the original data, these groups have large ratios, as compared with those adjacent, which means that the rate of mortality has been above the normal. Again, for entry ages 25 to 29 there is a large positive deviation, and the original data show a small ratio, meaning that at these ages the mortality has been unduly light. So again for entry ages 40 to 44, and for entry ages 50 to 54, where there are large deviations. See also pars. 39 and 40 above. The word "large" as applied to the deviations must not be misunderstood, because the deviations really bear but small proportions to the numbers of actual deaths in the several age groups. It is satisfactory thus to have it demonstrated that the grading method of construction has smoothed away the irregularities of the data, both perpendicular and horizontal, and that the new tables appear to interpret with very considerable accuracy the real meaning of the Experience.

47. For the British Offices' Experience the $O^{M(5)}$ Table has been used throughout as the ultimate; but, in the discussion which followed the reading of the former paper, one of the speakers suggested that it would have been better to employ the $O^{M(10)}$. I venture to say, however, that that would not have suited the purpose in view, which was to construct a select table with five years in the select period; so that necessarily it would have to be joined on to an ultimate table giving the mortality for lives assured for five years or more. It would not have been suitable under these circumstances to use an ultimate table excluding ten years of assurance. This is

from the theoretical point of view; but, even supposing the $O^{M(10)}$ Table had been used as ultimate, the only effect would have been slightly to alter the factors. There would have been no appreciable change in the rates of mortality brought out for the select columns.

48. Some curious questions arise regarding the two graduated ultimate tables of the British Offices' Experience, the $O^{M(5)}$ and the $O^{M(10)}$. The data for the $O^{M(5)}$ are given on pages 488 and 489 in the volume of "Assurance Experience." All duplicates have been eliminated as far as possible, and no individual life is included in any part of the table more than once. The late Sir G. F. Hardy based his graduation on the data thus, as it were, purified, using Makeham's law, with the constants $c = 1{\cdot}0939564$, $s = {\cdot}9941287$, and $g = {\cdot}9988449$. The results are excellent, from almost the youngest age to nearly the oldest, and there is hardly a trace of distortions, and the deviations of the expected deaths from the actual throughout the really important part of the table are never greater than must be produced in efficient graduation, no matter by what formula. Nothing could be better. See Hardy's comparison on page 149 of the volume of "Principles and Methods." Similar data are given for the $O^{M(10)}$ Table on pages 494 and 495 of the volume of Experience, and here again there are no duplicates, and this may be called the 1st edition of the $O^{M(10)}$ Table. It is identical in principle with the $O^{M(5)}$; but, for that very reason, it cannot strictly follow Makeham's law throughout. After the influence of new entrants has ceased, both tables are based on precisely the same materials. This condition is reached for all practical purposes at about attained age 75, and thereafter there is no appreciable difference between the tables. From that point onwards the two tables have the same Makeham constants, and, seeing that in the $O^{M(5)}$ these constants run from beginning to end, it is a necessary consequence that they do not apply strictly to the earlier portions of the 1st edition of the $O^{M(10)}$, although, as it happens, the difference is not great.

49. Hardy was of course aware of this elementary fact; and, seeing that he wanted a Makeham table as his ultimate for the select experience, he proceeded to construct a 2nd edition of the $O^{M(10)}$. His explanations, which are not easy to follow, are given in the volume of "Principles and Methods", commencing on page 146, and the 2nd edition of $O^{M(10)}$ materials appears on page 154. It would seem that he recast in some way not fully

explained the Select data given on pages 2 to 243 of the Experience volume, from which have been eliminated only duplicates arising from simultaneous policies on any individual life, retaining duplicates arising from successive policies on the same life; and from these materials he prepared his graduated table, having for Makeham constants $c = 1{\cdot}0939564$, $s = {\cdot}9940058$, and $g = {\cdot}9988579$. On the basis he adopted, the results are fairly good, the comparison of expected with actual deaths being given on page 155 of the "Principles and Methods." The extent to which duplicates are retained in the 2nd edition of the $O^{M(10)}$ may be measured from the following figures referring to ages 25 to 94. They seem to show that the duplicates amount to 22·6 per-cent of the actual number of lives exposed to risk, and to 26·7 per-cent of the actual deaths:

	At Risk	Deaths
Hardy's 2nd Edition	4,801,250	142,822
Experience 1st Edition	3,914,612	112,724
Duplicates	886,638	30,098

50. In order to bring out the actual effects that have been produced by the reconstruction of the $O^{M(10)}$ Table by Hardy, two comparisons are made in the small tables A and B, given below. In the first, the sums of five values of q_x for certain quinary age groups are taken, and in the second, the expected deaths, produced by multiplying Hardy's $O^{M(10)}$ q_x into the exposed to risk in the 1st edition of the $O^{M(10)}$ data, are compared with the actual deaths. For the first comparison, the sum of five values of q is taken instead of only one value, because at the older ages in both the $O^{M(5)}$ and the $O^{M(10)}$ official tables the q's run irregularly on account of their having been based on the column of l_x cut down to less than five figures. Attention was called to this in par. 42 of the former paper. The tables compared are the $O^{M(5)}$ and the $O^{M(10)}$ by Hardy, and the 1st edition of the $O^{M(10)}$ Table graduated by my own formula duplicated, which gives a very smooth curve, and clings closely to the original facts.* It will be seen that, except at the younger ages, Hardy's $O^{M(10)}$ Table shows a lighter mortality than does his $O^{M(5)}$, and it would seem from these figures as if the mortality

* *J.I.A.*, vol. xliii, Addendum, pars. 77 to 79, p. 169.

decreased with the duration of assurance, but that of course cannot be; and the conclusion is inevitable that Hardy's $O^{M(10)}$ Table understates the mortality. It is also seen that the 1st edition of the $O^{M(10)}$ Table shows a heavier mortality than does the $O^{M(5)}$, and that is what we should expect. The difference however, is not so great as might have been anticipated. In the second of the tables it is shown that almost throughout its whole extent Hardy's table persistently gives a lighter mortality than that of the 1st edition of the $O^{M(10)}$, and that the expected deaths by his rates of mortality applied to the 1st edition of the exposed to risk are about 1 per-cent deficient. It has been previously pointed out (see par. 39) that also during the select period the official table somewhat understates the mortality for entry ages 45 and over.

A.

Comparison of Σq_x

Ages	$O^{M(5)}$ Hardy	$O^{M(10)}$ Hardy	$O^{M(10)}$ King
30 to 34	·03898	·03948	·03933
40 „ 44	·05294	·05328	·05436
50 „ 54	·08705	·08702	·08789
60 „ 64	·16976	·16881	·16898
70 „ 74	·36689	·36380	·36884
80 „ 84	·81677	·80923	·82094

B.

Expected Deaths by Hardy's $O^{M(10)}q$, multiplied into E of $O^{M(10)}$ 1st Edition

Ages	Actual Deaths	Expected Deaths	Deviation
25 to 34	944	949·0	+ 5·0
35 „ 44	8485	8269·3	− 215·7
45 „ 54	18884	18702·7	− 181·3
55 „ 64	29051	28878·5	− 172·5
65 „ 74	32451	32048·7	− 402·3
75 „ 84	19352	19028·4	− 323·6
85 „ 94	3557	3699·1	+ 142·1
Total	112724	111575·7	−1148·3

51. When in the former paper attention was called to the introduction of irregularities into the rates of mortality at the older ages of the $O^{M(5)}$ Table, in the discussion which followed

it was pointed out that I myself had been guilty of the same offence in the case of the *Text-Book* Table, and that there the rates of mortality were also irregular at the older ages. I may here give the reason for the course which was followed, although perhaps some may not accept it as a valid defence. The *Text-Book* Table was constructed in about the year 1883, and at that time the only function that was ever used for graduation purposes was l_x. Makeham himself had used l_x with his own formula, as also Woolhouse with his summation formula. Naturally, therefore, that was the function used for the construction of the *Text-Book* Table. It was the column of l_x that was graduated, and not any other; and the object in view was, to bring the graduated column of l_x as near as possible to the ungraduated column. Also, the *Text-Book* Table was to be a guide to the student, and he was instructed that $\log p_x = \Delta \log l_x$, and that relationship is actually placed at the head of the columns of the Table. It might have confused the student if he had found that in parts of the table itself the relationship did not exist.

52. Only the first five years of the select period of the British Offices' Experience have been used to illustrate the methods of construction which have been explained. That was sufficient for the purpose, and it would have caused prohibitive labour to have tried to deal with all the ten. It is perhaps rash to say much about the columns from the sixth to the tenth inclusive without making tests, for which at present I cannot afford the time; but I may be permitted to venture on a few remarks as to the direction in which the indications afforded by the first five columns seem to point.

53. The first column of the select period has a character of its own, because the initial selection by the company greatly reduces the rate of mortality, and the adverse selection due to withdrawals has not yet come into play. The influence of the initial selection gradually diminishes as years pass, and diseases develop among the assured lives which run a more or less protracted course; but probably it is never completely lost, because the initial selection shuts out persons of weak physique, and members of short lived families. However that may be, the adverse effects of withdrawals begin after the first year, and continue indefinitely; and it seems to me that the general results are likely to become more uniform after a few years, and that there will be greater similarity between the columns.

54. I think therefore that all the five columns from the sixth to the tenth might very well be graded together. They would of course have to be first examined, and, if necessary, adjusted in the manner explained in pars. 8 and 9; and also adjusted so as to harmonize with the fifth column, but leaving that unaltered. They would then be examined by means of such a table as No. 1 appended, so as to decide the number of sections into which they should be divided. Then, after grading, the columns of factors would be constructed.

55. It would be useful if some one would take up this further enquiry, and I have great confidence that the results would be satisfactory. It may seem strange to some if I add that it would be quite suitable to use the $O^{M(5)}$ Table as the ultimate for the second five select columns, and so make the work on them strictly comparable with what I myself have done. The ultimate table used does not affect the graduated rates of mortality in the select columns. It affects only the factors.

56. Except for the Pension Fund for Nurses' rather freakish tables, select annuity tables have not so far been included in the investigations; and I think it would be very useful if some enterprising young actuary, who is not afraid of a little work, would take the question up. The selection of annuitants differs very much from that of assured lives. There is the initial self-selection against the companies at the purchase of annuities; but thereafter there is no disturbance from withdrawals. I think, therefore, that annuity tables should be easily and satisfactorily graduated by the new process, and that the select period would be a comparatively short one. Either the latest Government Annuity Tables, or those of the British Offices, might be used for investigation purposes.

57. Just as in the case of the British Offices' Experience, many experiments were tried on that of the AMP Society, but these will not be dealt with here, as a paper is in preparation for the Actuarial Society of Australasia, where the matter will be gone into. Only one of the graduations made is now submitted, in order to show that the new method gives good results even when applied to experiences of very diverse characters.

58. As compared with the $O^{M(5)}$ Table, the Australian Ultimate Table is very special. The mortality in Australia is much lighter than in the United Kingdom, and especially so at the younger ages. At age 20 in the ultimate table the

Australian mortality is only 60·5 per-cent of the home; at age 30, 62·6 per-cent; at age 40, 71·7 per-cent; at age 50, 75 per-cent; and at age 60, 77·6 per-cent; and at higher ages greater equality is reached between the two tables. The $(AMP)^{M(5)}$ Table has therefore a very much steeper gradient in the mortality curve than has the $O^{M(5)}$, and that greatly affects the factors that are used in the graduations. Moreover, in Australia there is not so much room for gain in selection as at home, on account of the already low mortality disclosed by the ultimate table. Therefore, the Australian factors used in the graduation are larger than the home ones, and they do not progress in the same way, so that the curves representing the columns of factors are very different.

59. The following are illustrations of the factors taken from the $(AMP)^{[M]}$ and the $O^{[M]}$ graduations respectively, which have been adopted as the best. They are taken in each case for the first year of the select period, which is not graded, and for the fifth year, which has been graded four years:

Factors.

Entry Age	1st Year		5th Year	
	$_1F$		$_5F$	
	$(AMP)^{[M]}$	$O^{[M]}$	$(AMP)^{[M]}$	$O^{[M]}$
20	·68651	·39508	·96174	·79722
30	·69678	·42346	·88858	·80169
40	·62525	·45183	·85069	·81869
50	·53080	·48021	·84593	·83121
60	·47235	·50858	·87212	·82225

60. It does not follow, however, that there is the same difference in the rates of mortality, because, within limits, the rates of mortality are not affected much by the ultimate table used, a change in the ultimate table causing a change in the factors mainly. The following are specimens of the rates of mortality corresponding to the factors given above:

Rates of Mortality.

Entry Age	$10^5 q_{[x]}$		$10^5 q_{[x]+4}$	
	(AMP)$^{[M]}$	O$^{[M]}$	(AMP)$^{[M]}$	O$^{[M]}$
20	270	258	398	542
30	326	316	487	654
40	438	442	711	938
50	615	742	1268	1626
60	1071	1486	2654	3217

61. From Table I it will be seen that the Australian mortality for entry ages 60 to 64 is very heavy indeed after the first year of the select period, and quite out of proportion to the mortality for the entry ages shown above it. This disturbs after about entry age 55 a table constructed from 48 entry ages; and it is evident that that range is too long for the AMP, and that the entry ages from 61 to 64 should be excluded. For the whole of the five years of the select period, and for all these four entry ages, there are only 28 deaths; so that 28 deaths have to be spread over twenty years of life. It is not to be wondered at, therefore, that the data for these ages are untrustworthy.

62. The fact that the range of 48 ages is too long is brought out also by the following comparison, which gives the ratios, graded in the way illustrated in Table IV, for the second year of the select period, being the first graded year:

Range.

—	48 ages	44 ages
${}_2R_1$	·83020	·82922
${}_2R_2$	·77205	·76737
${}_2R_3$	·70427	·74240
${}_2R_4$	·85485	·75179
${}_2R_\infty$	·77516	·77069

63. A factor uniform for each column gives fair results for both the experiences used as illustrations, and it follows that, whatever combinations are used in the grading, there should be nothing like violent fluctuations in the proportions of the ratios. When, however, the range of 48 ages is taken, the ratio

for the third section is very much smaller than for the second or the fourth, and that ought not to be. This is corrected when a range of only 44 ages is taken, and that shows that the 48 ages range is too long. Therefore, for the AMP, a range of only 44 ages was taken, divided into four sections of 11 ages each.

64. Before proceeding with the graduation, the AMP columns were adjusted as explained in par. 9 above, a transfer of fourteen deaths having been made from the 3rd column to the 4th, and of two deaths from the 4th column to the 5th.

65. A summary of the AMP graduation on these lines is given in Table VIII, and it may be well here to record the initial factor, and its differences, used for the table. The first year was treated by itself, and we have :

$$ {}_1F_1 = \cdot 6586686; \quad \delta = +\cdot 0107855 $$
$$ \delta^2 = -\cdot 0015249; \quad \delta^3 = +\cdot 0000589 $$

The remaining four years were graded together, and for the 2nd year, which is the first graded year, the figures are as follows :

$$ {}_2F_1 = \cdot 8685305; \quad \delta = -\cdot 0088502 $$
$$ \delta^2 = +\cdot 0003321; \quad \delta^3 = -\cdot 0000019 $$

As to the years from the 3rd to the 5th inclusive, they were dealt with as explained in par. 35 above.

66. Taken as a whole, the graduation now submitted seems to be satisfactory, and to give a good interpretation of the original data ; and I think it may be used with entire confidence for practical work. The deviations in the last section of the table, that for all the five years of the select period, are not very small as compared with the number of deaths, but they are well distributed, and the comparative magnitude of these deviations is due to irregularities in the original data, the AMP Experience being much rougher than that of the British Offices.

67. This can be brought out for the AMP as was done in par. 45 above for the British Offices ; and the following are the average ratios for the five years of the select period, (*a*) by the original data, and (*b*) by the graduated table.

Entry Ages	Average Ratio for the Five Years of the Select Period	
	(*a*)	(*b*)
17 to 19	·71968	·85226
20 ,, 24	·84030	·83591
25 ,, 29	·87704	·81665
30 ,, 34	·78918	·79635
35 ,, 39	·74863	·77342
40 ,, 44	·83641	·75709
45 ,, 49	·64410	·74765
50 ,, 54	·71032	·74362
55 ,, 59	·77146	·74542
60	·92392	·75000

It will be seen that wherever the ratio under (*a*) is unduly small, there is a positive deviation in Table VIII, and wherever the ratio under (*a*) is unduly large, there is a negative deviation, showing that the original data are at fault, and not the graduation.

68. In the discussion which followed the reading of the former paper Mr. Kenchington put forward a most valuable suggestion. It was admitted in the paper that the uniform factor for each column is only an approximation, which could be improved upon, although it may give good results in some cases ; and it was shown further that, in the case of the British Offices' Experience, the uniform factor had the tendency to understate the mortality at the older entry ages. To remedy this defect, Mr. Kenchington proposed that the uniform factor should be combined with a constant, to be deducted from the results of multiplying the ultimate q's by the factor. In this way the column of the ratios of the graduated deaths to the ultimate expected deaths becomes an increasing series, the increase, however, being at a diminishing rate with advancing entry age. The deduction of the constant has greater effect when q is small than when q is large ; and hence the form taken by the column of ratios. Mr. Kenchington himself tested his method on the British Offices' Experience for the whole of the ten years of the select period, and found it to be on the whole successful. He found, moreover, that in the later years of the select period the constant might be additive instead of subtractive.

69. Mr. Kenchington calculated the values of the factors and the constants by means of two equations, based on two

summations of the data. The process may be explained symbolically as follows: Let E represent the exposed to risk, D the ultimate expected deaths, and θ the actual deaths. Also, let R be the factor and K the constant. The column is divided into three sections of equal lengths, and we have E_1 for the exposed to risk in the first section, E_2 in the second, and E_3 in the third: and similarly for D and θ.

70. Proceeding:

First Summation.	*Second Summation.*
E_1	E_1
E_2	E_1+E_2
E_3	$E_1+E_2+E_3$
$(\Sigma_1 E)=E_1+E_2+E_3$	$(\Sigma_2 E)=3E_1+2E_2+E_3$

Similarly for D and θ.

Then
$$(\Sigma_1 E)K+(\Sigma_1 D)R=(\Sigma_1\theta)$$
$$(\Sigma_2 E)K+(\Sigma_2 D)R=(\Sigma_2\theta)$$

Whence K and R can be found.

71. The values of R and K can be found otherwise, by dividing the column into only two sections, and treating them separately. In this way,

$$E_1K+D_1R=\theta_1 \text{ and } E_2K+D_2R=\theta_2.$$

The two methods produce results which differ but little. Taking the first select year of the $O^{[M]}$ Experience, with a range of 48 entry ages from 17 to 64 inclusive, and, for reasons set forth above, retaining the $O^{M(5)}$ Table as the ultimate, the summation method gives $R=\cdot56065$ and $K=-\cdot00108$; whereas the two independent sections method gives $R=\cdot56318$, and $K=-\cdot00110$.

72. The Kenchington method is very easy to employ; and in fact, the amount of work required is hardly more than when a uniform factor is used. Moreover, it has the further great advantage that, when the values of R and K have been calculated, the graduation can be tested before the complete column of q' is constructed. For any age group the relationship exists (the graduated expected deaths being written d), $RD+KE=d$; so that with a minimum of labour, such a summary table as No. IX annexed can be prepared.

73. The Kenchington method gives a good perpendicular graduation in suitable cases; but, when it is applied to the successive columns of a select table separately, it breaks down, just as does the method which may be called "The Finite Differences Method." The irregularities as between the columns have to be removed by horizontal graduation, and that can be done by the method of grading. Therefore, in suitable cases, the Kenchington method, supplemented by grading, supplies an excellent short method of constructing select mortality tables.

74. A graduation of the British Offices' Experience effected in this way is summarized in Table IX; and this should be compared with Table VI, which gives a summary of the graduation by the Finite Differences method. Also, the following are specimens of the Kenchington rates of mortality, and these should be compared with the corresponding rates in Table V, (A) to (E). It will be seen that the two graduations are not far from being identical.

$O^{[M]}$ *Experience.*

Entry Age	Rates of Mortality by the Kenchington Method					Entry Age
	1st Year	2nd Year	3rd Year	4th Year	5th Year	
20	·00257	·00422	·00483	·00519	·00540	20
25	·00278	·00448	·00517	·00559	·00584	25
30	·00310	·00492	·00570	·00620	·00655	30
35	·00361	·00559	·00654	·00718	·00764	35
40	·00440	·00665	·00785	·00868	·00934	40
45	·00565	·00829	·00989	·01105	·01202	45
50	·00760	·01087	·01307	·01475	·01619	50
55	·01063	·01488	·01805	·02050	·02266	55
60	·01535	·02112	·02576	·02944	·03271	60

75. The Kenchington method, however, overstates slightly the mortality at the older entry ages; and, where the results of the two graduations differ, I venture to think that those of the Finite Differences method should be preferred. It is the more pliable of the two, and fits better into any undulations that should be retained in the column of factors.

76. The reason why the Kenchington method, when combined with the grading process, gives such good results in the case of the British Offices' Experience is, that the columns of factors have only shallow undulations of wide sweep, with small differences, and the method averages them. It is only after

entry age 52 in the graded columns, where there is a maximum point, and the differences become comparatively large, that the error involved in such averaging reaches any magnitude. On this point see par. 41 above.

77. The Kenchington method is not universally applicable. For instance, it does not suit the AMP Experience. This is seen when the columns of factors are examined. The following are the factors at quinquennial intervals for the first year, which is graduated independently, and for the second year, which is the first of the graded years:

(AMP)[M] *Factors.*

Entry Age	1st Year—Not Graded		2nd Year - Graded	
	Factor	Δ	Factor	Δ
20	·68651	+1904	·84297	−3605
25	·70555	− 877	·80692	−2808
30	·69678	−2922	·77884	−2035
35	·66756	−4231	·75849	−1286
40	·62525	−4804	·74563	− 560
45	·57721	−4641	·74003	+ 143
50	·53080	−3741	·74146	+ 820
55	·49339	−2104	·74966	+1475
60	·47235	...	·76441	...

The differences are always large, and in the column for the first year there is a maximum point at entry age 26; while, in the columns for the four graded years, there is a minimum point at entry age 46.

78. It may be useful to record the ratios and the constants which were used in the graduation summarized in Table IX. They are as follows:

Year	R	K
1st	·5631823	−·0011033
2nd	·6819807	−·0002773
3rd	·7728314	−·0003076
4th	·8194612	−·0003206
5th	·8449686	−·0003418

79. A good idea as to whether the Kenchington method is

applicable to any particular case can be formed by studying such a table as No. I appended.

SCHEME OF NOTATION.

E = Exposed to Risk.

θ = Actual Deaths.

D = Ultimate Expected Deaths.

d = Graduated Expected Deaths.

q = Ultimate Rate of Mortality.

q' = Rate of Mortality in the Select Period.

R = Ratio of θ to D; so that $R=\theta\div D$.

r See explanations below.

A suffix to the *left* of a symbol denotes the column of the Select Period to which the symbol belongs; and a suffix to the *right* of a symbol denotes the position in the column. Thus, ${}_3F_{20}$ means the twentieth factor down the column for the third year after selection.

If a column be divided into sections, the suffix to the right of a symbol denotes the number of the section. Thus, ${}_3R_4$ means the ratio of actual deaths to ultimate expected deaths for the aggregate of the ages in the fourth section of the column for the third year after selection. Thus, for example, we have the equation ${}_3R_4={}_3\theta_4\div{}_3D_4$.

If the whole column be in question, everything in the column being included, the symbol ∞ is employed for the suffix; and we have ${}_3R_\infty={}_3\theta_\infty\div{}_3D_\infty$. It will be noticed that this represents the ratio used in the former paper as a uniform factor for the whole column.

In the grading process all the columns in the select period, or a given number of them, are combined, and the whole of the data in the graded columns is brought into account. This is denoted by the suffix ∞ being placed to the left of the symbol. Thus we have ${}_\infty R_\infty={}_\infty\theta_\infty\div{}_\infty D_\infty$, as in Table III.

The symbol r denotes the proportion which the ratio for the total of a section bears to the ratio for the total of all the sections combined. Thus we have in Table III $r_1={}_\infty R_1\div{}_\infty R_\infty$; and so on. In this connection see par. 26, and also Table III.

Table I.

Ratios for Age Groups of the Actual Deaths to the Ultimate Expected Deaths.

$O^{[M]}$.

Entry Ages	1st Year	2nd Year	3rd Year	4th Year	5th Year	Entry Ages
17 to 19	·4687	·7479	·6407	·8675	·9500	17 to 19
20 ,, 24	·3863	·7007	·7982	·8030	·8346	20 ,, 24
25 ,, 29	·3947	·5784	·7028	·7635	·8059	25 ,, 29
30 ,, 34	·4566	·6124	·7429	·7608	·8281	30 ,, 34
35 ,, 39	·4081	·6629	·7496	·7960	·8186	35 ,, 39
40 ,, 44	·4905	·6323	·7326	·7921	·7528	40 ,, 44
45 ,, 49	·4387	·6795	·7069	·8632	·8375	45 ,, 49
50 ,, 54	·5188	·7626	·8046	·8091	·8103	50 ,, 54
55 ,, 59	·5110	·7240	·7692	·7668	·7530	55 ,, 59
60 ,, 64	·5104	·6902	·7342	·6873	·9197	60 ,, 64
17 to 64	·4430	·6529	·7417	·7882	·8130	17 to 64

$(AMP)^{[M]}$.

Entry Ages	1st Year	2nd Year	3rd Year	4th Year	5th Year	Entry Ages
17 to 19	·6897	·9009	·7921	·3333	·8642	17 to 19
20 ,, 24	·7348	·7097	·7529	1·1705	·9341	20 ,, 24
25 ,, 29	·6787	·9140	·9798	·7622	1·1397	25 ,, 29
30 ,, 34	·6999	·8161	·7692	·8447	·8431	30 ,, 34
35 ,, 39	·5931	·6495	·8929	·8378	·8183	35 ,, 39
40 ,, 44	·7181	·8716	·7924	·9984	·8277	40 ,, 44
45 ,, 49	·4461	·6135	·9544	·5252	·7051	45 ,, 49
50 ,, 54	·5483	·7324	·8215	·6479	·8146	50 ,, 54
55 ,, 59	·4525	·7246	·7282	1·0732	·9091	55 ,, 59
60 ,, 64	·5376	1·0227	1·7241	·9639	·9756	60 ,, 64
17 to 64	·6380	·7752	·8640	·8337	·8764	17 to 64

TABLE II.

$O^{[M]}$ *Experience.* *Entry Ages* 17 *to* 64.

Viewed Perpendicularly

2ND YEAR		3RD YEAR	
Ratio	Proportion	Ratio	Proportion
R_1 =.62487	·9571	R_1 =·73192	·9869
R_2 =·63856	·9781	R_2 =·74148	·9997
R_3 =·67540	1·0345	R_3 =·74021	·9980
R_4 =·71429	1·0941	R_4 =·76555	1·0322
R_∞ =·65287	...	R_∞ =·74169	...

4TH YEAR		5TH YEAR	
Ratio	Proportion	Ratio	Proportion
R_1 =·76631	·9721	R_1 =·82684	1·0171
R_2 =·78472	·9955	R_2 =·81861	1·0070
R_3 =·82989	1·0528	R_3 =·79900	·9829
R_4 =·74811	·9491	R_4 =·80066	·9849
R_∞ =·78824	...	R_∞ =·81297	...

Viewed Horizontally

1ST SECTION			2ND SECTION		
Year	Ratio R_1	Rate of Progression	Year	Ratio R_2	Rate of Progression
2nd	·62487	1·1713	2nd	·63856	1·1612
3rd	·73192	1·0470	3rd	·74148	1·0584
4th	·76631	1·0790	4th	·78472	1·0432
5th	·82684	...	5th	·81861	...

3RD SECTION			4TH SECTION		
Year	Ratio R_3	Rate of Progression	Year	Ratio R_4	Rate of Progression
2nd	·67540	1·0960	2nd	·71429	1·0718
3rd	·74021	1·1215	3rd	·76555	·9773
4th	·82989	·9628	4th	·74811	1·0702
5th	·79900	...	5th	·80066	...

Graded Ratios. Columns for Years 2 *to* 5 *Combined.*

PERPENDICULARLY		HORIZONTALLY		
Ratio	Proportion	Year	Ratio	Rate of Progression
R_1 =·73416	·9813	2nd	·65287	1·1361
R_2 =·74462	·9952	3rd	·74169	1·0628
R_3 =·76154	1·0179	4th	·78824	1·0314
R_4 =·75789	1·0130	5th	·81297	...
R_∞ =·74818	...	...	...	...

TABLE III.

$O^{[M]}$ *Experience.* *Entry Ages* 17 *to* 64. *Uniform Grading.*
4 *Sections of* 12 *Ages each.* 4 *Years Graded.*

Year	Actual Deaths	Ultimate Expected Deaths	
1st Section.—Entry Ages 17 to 28.			
2nd 3rd 4th 5th	615 682 687 721	984·2 931·8 896·5 872·0	${}_{\infty}R_1 = \cdot 7341566$ $r_1 = \cdot 9812598$
...	2705	3684·5	$= {}_{\infty}R_1 \div {}_{\infty}R_{\infty}$
2nd Section.—Entry Ages 29 to 40.			
2nd 3rd 4th 5th	1080 1219 1274 1321	1691·3 1644·0 1623·5 1613·7	${}_{\infty}R_2 = \cdot 7446177$ $r_2 = \cdot 9952419$
	4894	6572·5	$= {}_{\infty}R_2 \div {}_{\infty}R_{\infty}$
3rd Section.—Entry Ages 41 to 52.			
2nd 3rd 4th 5th	712 779 883 865	1054·2 1052·4 1064·0 1082·6	${}_{\infty}R_3 = \cdot 7615442$ $r_3 = 1 \cdot 0178655$
	3239	4253·2	$= {}_{\infty}R_3 \div {}_{\infty}R_{\infty}$
4th Section.—Entry Ages 53 to 64.			
2nd 3rd 4th 5th	325 353 354 392	455·0 461·1 473·2 489·6	${}_{\infty}R_4 = \cdot 7578903$ $r_4 = 1 \cdot 0129818$
	1424	1878·9	$= {}_{\infty}R_4 \div {}_{\infty}R_{\infty}$
Total	12262	16389·1	${}_{\infty}R_{\infty} = \cdot 7481776$

Table IV.

O[M] *Experience.* *Entry Ages* 17 *to* 64. *Uniform Grading.*
4 *Sections of* 12 *Ages each.* 4 *Years Graded.*

Entry Ages	Actual Deaths	Ultimate Ex. Deaths	Crude Ratio	Graded Ratio	Graded Deaths
1st Year.—Not Graded.					
17 to 28	405	1067·5	·37939	...	...
29 ,, 40	785	1771·3	·44318	...	...
41 ,, 52	509	1072·0	·47482	...	...
53 ,, 64	234	452·7	·51691	...	...
17 to 64	1933	4363·5	·44299	...	...
2nd Year.					
17 to 28	615	984·2	·62487	·64064	630·5
29 ,, 40	1080	1691·3	·63856	·64976	1098·9
41 ,, 52	712	1054·2	·67540	·66453	700·5
53 ,, 64	325	455·0	·71429	·66135	300·9
17 to 64	2732	4184·7	·65287	·65287	2730·8
3rd Year.					
17 to 28	682	931·8	·73192	·72779	678·2
29 ,, 40	1219	1644·0	·74148	·73816	1213·5
41 ,, 52	779	1052·4	·74021	·75494	794·5
53 ,, 64	353	461·1	·76555	·75132	346·4
17 to 64	3033	4089·3	·74169	·74169	3032·6
4th Year.					
17 to 28	687	896·5	·76631	·77347	693·4
29 ,, 40	1274	1623·5	·78472	·78449	1273·6
41 ,, 52	883	1064·0	·82989	·80232	853·7
53 ,, 64	354	473·2	·74811	·79847	377·8
17 to 64	3198	4057·2	·78824	·78824	3198·5
5th Year.					
17 to 28	721	872·0	·82684	·79773	695·6
29 ,, 40	1321	1613·7	·81861	·80910	1305·6
41 ,, 52	865	1082·6	·79900	·82749	895·8
53 ,, 64	392	489·6	·80066	·82352	403·2
17 to 64	3299	4057·9	·81297	·81297	3300·2

TABLE V (A).

British Offices' Select Table. First Year.

Entry Age $[x]$	Original Data: At Risk	Original Data: Actual Deaths	$O^{M(5)}$ Expected Deaths	Factor	$q'_{[x]}$	Graduated Expected Deaths
17	1682	7	10·7	·38657	·00247	4·1
8	2572	8	16·5	·38941	·00250	6·4
9	3713	9	24·0	·39224	·00253	9·4
20	5768	17	37·6	·39508	·00258	14·9
1	10010	26	66·0	·39792	·00262	26·3
2	12392	27	82·4	·40076	·00267	33·0
3	15099	31	101·5	·40359	·00271	41·0
4	17861	57	121·5	·40643	·00276	49·4
25	20401	50	140·6	·40927	·00282	57·5
6	21501	51	150·1	·41211	·00288	61·9
7	22024	69	156·2	·41494	·00294	64·8
8	22249	53	160·4	·41778	·00301	67·0
9	22606	82	165·5	·42062	·00308	69·6
30	23113	72	172·7	·42346	·00316	73·1
1	21215	76	161·7	·42629	·00325	68·9
2	20535	75	159·6	·42913	·00333	68·5
3	19670	67	156·6	·43197	·00344	67·7
4	18505	76	151·0	·43481	·00355	65·7
35	17790	71	148·9	·43764	·00366	65·2
6	16510	57	142·0	·44048	·00379	62·5
7	14944	52	132·4	·44332	·00393	58·7
8	14157	53	129·5	·44616	·00408	57·8
9	13337	44	126·0	·44899	·00424	56·6
40	12821	60	125·4	·45183	·00442	56·7
1	11235	42	114·0	·45467	·00461	51·8
2	10434	58	110·2	·45751	·00483	50·4
3	9517	61	104·6	·46034	·00506	48·1
4	8749	51	100·3	·46318	·00531	46·5
45	8202	46	98·4	·46602	·00559	45·9
6	7152	43	89·8	·46886	·00589	42·1
7	6466	31	85·4	·47169	·00623	40·3
8	5860	32	81·3	·47453	·00659	38·6
9	5346	38	78·2	·47737	·00698	37·3
50	5193	49	80·2	·48021	·00742	38·5
1	4045	21	66·1	·48304	·00789	31·9
2	3670	37	63·5	·48588	·00841	30·9
3	3193	27	58·7	·48872	·00899	28·7
4	2825	34	55·3	·49156	·00961	27·2
55	2467	17	51·4	·49439	·01030	25·4
6	2110	22	46·9	·49723	·01105	23·3
7	1700	23	40·4	·50007	·01188	20·2
8	1477	23	37·5	·50291	·01278	18·9
9	1363	24	37·1	·50574	·01377	18·8
60	1348	25	39·4	·50858	·01486	20·0
1	806	12	25·3	·51142	·01605	12·9
2	712	12	24·0	·51426	·01735	12·3
3	541	10	19·6	·51709	·01878	10·2
4	438	5	17·1	·51993	·02034	8·9

TABLE V (B).

British Offices' Select Table. Second Year.

Age		Original Data		$^{O}M^{(5)}$ Expected Deaths	Factor	$q'_{[x]+1}$	Graduated Expected Deaths
Entry $[x]$	Attnd. $[x]+1$	At Risk	Actual Deaths				
17	18	1553	7	10·0	·64315	·00413	6·4
8	9	2303	9	14·9	·64192	·00415	9·6
9	20	3361	19	21·9	·64094	·00418	14·0
20	21	5128	30	33·8	·64022	·00422	21·6
1	2	8939	44	59·4	·63974	·00425	38·0
2	3	11182	49	75·1	·63948	·00430	48·0
3	4	13603	57	92·5	·63943	·00435	59·1
4	25	16215	81	111·7	·63958	·00441	71·4
25	26	18530	81	129·3	·63990	·00447	82·7
6	7	19684	71	139·6	·64040	·00454	89·4
7	8	20192	97	145·6	·64106	·00462	93·3
8	9	20540	70	150·4	·64185	·00470	96·5
9	30	20901	98	156·1	·64277	·00480	100·3
30	31	21401	93	163·1	·64381	·00491	105·0
1	2	19698	89	153·1	·64495	·00501	98·7
2	3	19072	89	151·8	·64617	·00514	98·1
3	4	18254	106	149·0	·64747	·00528	96·5
4	35	17195	89	143·9	·64883	·00543	93·4
35	36	16615	89	142·9	·65023	·00559	92·9
6	7	15391	89	136·4	·65167	·00577	88·9
7	8	13888	94	127·1	·65312	·00598	83·0
8	9	13183	82	124·6	·65458	·00619	81·6
9	40	12494	79	122·2	·65603	·00642	80·2
40	41	11928	83	121·1	·65746	·00667	79·6
1	2	10478	65	110·6	·65885	·00696	72·9
2	3	9734	65	107·0	·66020	·00726	70·6
3	4	8946	65	102·5	·66148	·00758	67·8
4	45	8174	63	98·1	·66268	·00795	65·0
45	46	7710	71	96·8	·66379	·00834	64·3
6	7	6690	57	88·3	·66480	·00878	58·7
7	8	6062	60	84·1	·66569	·00924	56·0
8	9	5479	53	80·2	·66645	·00975	53·4
9	50	5012	49	77·4	·66706	·01031	51·6
50	51	4908	62	80·2	·66752	·01091	53·5
1	2	3797	51	65·7	·66780	·01156	43·9
2	3	3440	51	63·3	·66790	·01228	42·3
3	4	2964	46	58·0	·66779	·01306	38·7
4	55	2660	36	55·4	·66748	·01390	37·0
55	56	2312	33	51·4	·66693	·01482	34·3
6	7	1980	41	47·0	·66615	·01582	31·3
7	8	1583	29	40·2	·66511	·01690	26·7
8	9	1395	28	38·0	·66380	·01807	25·2
9	60	1283	24	37·5	·66221	·01934	24·8
60	61	1262	25	39·6	·66032	·02072	26·1
1	2	762	19	25·7	·65813	·02220	16·9
2	3	673	23	24·4	·65561	·02381	16·0
3	4	512	10	20·0	·65276	·02554	13·1
4	65	421	11	17·8	·64955	·02742	11·6

TABLE V (C).

British Offices' Select Table. Third Year.

Age: Entry $[x]$	Age: Attnd. $[x]+2$	Original Data: At Risk	Original Data: Actual Deaths	$O^{M(5)}$ Expected Deaths	Factor	$q'_{[x]+2}$	Graduated Expected Deaths
17	19	1450	7	9·4	·73065	·00472	6·9
8	20	2116	7	13·8	·72925	·00475	10·1
9	1	3113	14	20·5	·72814	·00480	14·9
20	22	4694	25	31·2	·72732	·00484	22·7
1	3	8194	45	55·1	·72678	·00488	40·0
2	4	10357	70	70·4	·72648	·00494	51·1
3	25	12571	69	86·6	·72642	·00501	62·9
4	6	15045	69	105·0	·72659	·00507	76·3
25	27	17368	79	123·1	·72696	·00515	89·5
6	8	18382	87	132·5	·72753	·00525	96·4
7	9	18993	117	139·0	·72828	·00533	101·2
8	30	19442	93	145·2	·72917	·00545	105·9
9	1	19726	109	150·3	·73022	·00556	109·8
30	32	20081	111	156·0	·73140	·00568	114·1
1	3	18608	110	148·1	·73270	·00583	108·5
2	4	18044	122	147·2	·73408	·00599	108·1
3	35	17239	104	144·3	·73556	·00616	106·1
4	6	16205	99	139·4	·73710	·00634	102·8
35	37	15728	115	139·4	·73869	·00654	103·0
6	8	14593	88	133·5	·74033	·00677	98·8
7	9	13134	88	124·1	·74198	·00701	92·1
8	40	12453	95	121·8	·74364	·00727	90·6
9	1	11838	93	120·2	·74528	·00756	89·6
40	42	11332	85	119·7	·74691	·00789	89·4
1	3	10013	72	110·0	·74849	·00823	82·3
2	4	9276	86	106·3	·75002	·00860	79·7
3	45	8463	71	101·6	·75147	·00902	76·3
4	6	7763	78	97·5	·75284	·00946	73·4
45	47	7310	68	96·5	·75410	·00995	72·8
6	8	6338	65	88·0	·75525	·01048	66·5
7	9	5784	62	84·6	·75626	·01106	64·0
8	50	5184	57	80·1	·75712	·01170	60·6
9	1	4773	50	78·0	·75781	·01238	59·1
50	52	4639	78	80·3	·75834	·01313	60·9
1	3	3556	41	65·4	·75865	·01395	49·6
2	4	3278	51	64·1	·75877	·01484	48·6
3	55	2816	63	58·7	·75864	·01580	44·5
4	6	2515	28	55·9	·75829	·01685	42·4
55	57	2161	40	51·3	·75767	·01799	38·9
6	8	1873	37	47·6	·75678	·01923	36·0
7	9	1511	33	41·1	·75560	·02057	31·1
8	60	1335	28	39·0	·75411	·02203	29·4
9	1	1215	29	38·1	·75230	·02361	28·7
60	62	1181	31	39·8	·75016	·02530	29·9
1	3	731	21	26·5	·74767	·02716	19·8
2	4	626	17	24·5	·74481	·02914	18·2
3	65	487	11	20·6	·74157	·03130	15·3
4	6	395	15	18·0	·73792	·03360	13·3

TABLE V (D).

British Offices' Select Table. Fourth Year.

Age: Entry $[x]$	Age: Attnd. $[x]+3$	Original Data: At Risk	Original Data: Actual Deaths	OM(5) Expected Deaths	Factor	$q'_{[x]+3}$	Graduated Expected Deaths
17	20	1369	7	8·9	·77650	·00506	6·9
8	1	2008	10	13·2	·77502	·00511	10·2
9	2	2924	19	19·4	·77384	·00515	15·0
20	23	4365	24	29·3	·77297	·00519	22·6
1	4	7686	52	52·3	·77239	·00525	40·4
2	25	9713	59	66·9	·77207	·00532	51·7
3	6	11890	57	83·0	·77201	·00539	64·1
4	7	14239	75	101·0	·77219	·00547	78·0
25	28	16507	90	119·0	·77258	·00557	91·9
6	9	17489	93	128·0	·77318	·00566	99·0
7	30	18004	96	134·5	·77398	·00578	104·1
8	1	18508	105	141·0	·77494	·00591	109·3
9	2	18892	127	146·8	·77605	·00603	113·9
30	33	19191	105	152·8	·77730	·00619	118·8
1	4	17836	112	145·5	·77868	·00635	113·3
2	35	17220	94	144·1	·78015	·00653	112·4
3	6	16553	117	142·4	·78172	·00672	111·3
4	7	15587	122	138·1	·78336	·00694	108·2
35	38	14985	112	137·1	·78505	·00718	107·6
6	9	13967	108	132·0	·78679	·00744	103·9
7	40	12670	98	123·9	·78854	·00771	97·7
8	1	11970	89	121·5	·79030	·00802	96·0
9	2	11358	98	119·9	·79206	·00836	95·0
40	43	10861	92	119·4	·79378	·00872	94·8
1	4	9681	103	110·9	·79546	·00912	88·2
2	45	8890	89	106·7	·79709	·00957	85·0
3	6	8128	82	102·1	·79864	·01003	81·5
4	7	7480	60	98·7	·80008	·01056	79·0
45	48	7024	75	97·5	·80142	·01112	78·1
6	9	6052	76	88·5	·80264	·01174	71·0
7	50	5540	82	85·6	·80372	·01242	68·8
8	1	4956	69	81·0	·80464	·01315	65·2
9	2	4595	71	79·5	·80537	·01394	64·0
50	53	4409	72	81·1	·80593	·01482	65·3
1	4	3426	52	67·0	·80627	·01577	54·0
2	55	3138	52	65·4	·80639	·01680	52·7
3	6	2665	46	59·2	·80625	·01791	47·7
4	7	2413	45	57·3	·80588	·01914	46·2
55	58	2073	44	52·7	·80522	·02046	42·4
6	9	1814	33	49·4	·80427	·02189	39·7
7	60	1468	22	42·9	·80302	·02346	34·4
8	1	1291	38	40·5	·80144	·02515	32·5
9	2	1151	35	38·8	·79952	·02697	31·0
60	63	1116	32	40·5	·79723	·02896	32·3
1	4	691	27	27·0	·79459	·03108	21·5
2	65	590	9	24·9	·79155	·03341	19·7
3	6	472	15	21·5	·78811	·03589	16·9
4	7	377	8	18·5	·78423	·03857	14·5

Table V (E).

British Offices' Select Table. Fifth Year.

Age: Entry $[x]$	Age: Attnd. $[x]+4$	Original Data: At Risk	Original Data: Actual Deaths	$O^{M(5)}$ Expected Deaths	Factor	$q'_{[x]+4}$	Graduated Expected Deaths
17	21	1320	6	8·7	·80087	·00528	7·0
8	2	1901	11	12·6	·79933	·00532	10·1
9	3	2779	21	18·7	·79811	·00536	14·9
20	24	4144	22	28·2	·79722	·00542	22·5
1	25	7238	61	49·9	·79662	·00549	39·8
2	6	9218	56	64·3	·79630	·00556	51·2
3	7	11351	59	80·5	·79623	·00565	64·1
4	8	13618	70	98·2	·79642	·00574	78·2
25	29	15840	83	115·9	·79682	·00583	92·4
6	30	16774	107	125·3	·79744	·00596	99·9
7	1	17248	115	131·4	·79826	·00608	104·9
8	2	17796	110	138·3	·79925	·00621	110·5
9	3	18124	113	144·3	·80039	·00637	115·5
30	34	18496	121	150·9	·80169	·00654	121·0
1	35	17172	121	143·7	·80311	·00672	115·4
2	6	16591	112	142·7	·80463	·00692	114·8
3	7	15897	121	140·8	·80625	·00714	113·5
4	8	14953	117	136·8	·80794	·00739	110·5
35	39	14427	109	136·3	·80968	·00765	110·4
6	40	13438	113	131·4	·81148	·00794	106·6
7	1	12248	103	124·3	·81328	·00825	101·1
8	2	11528	88	121·7	·81510	·00861	99·2
9	3	10949	106	120·3	·81690	·00897	98·3
40	44	10511	97	120·5	·81869	·00938	98·7
1	45	9362	81	112·3	·82042	·00985	92·1
2	6	8573	81	107·7	·82210	·01033	88·5
3	7	7801	84	103·0	·82369	·01087	84·8
4	8	7193	66	99·8	·82519	·01145	82·4
45	49	6789	84	99·3	·82657	·01209	82·1
6	50	5814	76	89·8	·82783	·01279	74·3
7	1	5342	70	87·3	·82893	·01354	72·4
8	2	4787	69	82·9	·82988	·01437	68·8
9	3	4419	70	81·3	·83064	·01528	67·5
50	54	4242	69	83·0	·83121	·01626	69·0
1	55	3288	41	68·5	·83156	·01732	57·0
2	6	3048	74	67·7	·83169	·01848	56·3
3	7	2572	45	61·1	·83155	·01975	50·8
4	8	2327	46	59·1	·83116	·02112	49·1
55	59	2002	42	54·5	·83048	·02261	45·3
6	60	1743	42	50·9	·82951	·02423	42·7
7	1	1446	35	45·4	·82821	·02599	37·6
8	2	1233	32	41·6	·82658	·02788	34·4
9	3	1100	24	40·0	·82460	·02995	33·0
60	64	1062	30	41·5	·82225	·03217	34·1
1	65	660	22	27·9	·81952	·03459	22·9
2	6	575	27	26·2	·81638	·03718	21·4
3	7	448	23	22·0	·81283	·03997	17·9
4	8	365	24	19·4	·80884	·04301	15·7

TABLE VI.

$O^{[M]}$ *Experience. Entry Ages* 17 *to* 64.

1*st Year* 2 *Sections. Not Graded. Remaining Years* 4 *Sections Graded* 4 *Years.*

Entry Ages	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Entry Ages
	1st Year—Not Graded			2nd Year			3rd Year			
17 to 19	24	19·9	− 4·1	35	30·0	− 5·0	28	31·9	+ 3·9	17 to 19
20 ,, 24	158	164·6	+ 6·6	261	238·1	− 22·9	278	253·0	− 25·0	20 ,, 24
25 ,, 29	305	320·8	+ 15·8	417	462·2	+ 45·2	485	502·8	+ 17·8	25 ,, 29
30 ,, 34	366	343·9	− 22·1	466	491·7	+ 25·7	546	539·6	− 6·4	30 ,, 34
35 ,, 39	277	300·8	+ 23·8	433	426·6	− 6·4	479	474·1	− 4·9	35 ,, 39
40 ,, 44	272	253·5	− 18·5	341	355·9	+ 14·9	392	401·1	+ 9·1	40 ,, 44
45 ,, 49	190	204·2	+ 14·2	290	284·0	− 6·0	302	323·0	+ 21·0	45 ,, 49
50 ,, 54	168	157·2	− 10·8	246	215·4	− 30·6	261	246·0	− 15·0	50 ,, 54
55 ,, 59	109	106·6	− 2·4	155	142·3	− 12·7	167	164·1	− 2·9	55 ,, 59
60 ,, 64	64	64·3	+ 0·3	88	83·7	− 4·3	95	96·5	+ 1·5	60 ,, 64
Totals	1933	1935·8	+ 60·7 − 57·9	2732	2729·9	+ 85·8 − 87·9	3033	3032·1	+ 53·3 − 54·2	Totals
			± 118·6			± 173·7			± 107·5	
	4th Year			5th Year			All 5 Years			
17 to 19	36	32·1	− 3·9	38	32·0	− 6·0	161	145·9	− 15·1	17 to 19
20 ,, 24	267	256·8	− 10·2	268	255·8	− 12·2	1232	1168·3	− 63·7	20 ,, 24
25 ,, 29	511	518·2	+ 7·2	528	523·2	− 4·8	2246	2327·2	+ 81·2	25 ,, 29
30 ,, 34	550	564·0	+ 14·0	592	575·2	− 16·8	2520	2514·4	− 5·6	30 ,, 34
35 ,, 39	505	500·2	− 4·8	519	515·6	− 3·4	2213	2217·3	+ 4·3	35 ,, 39
40 ,, 44	426	428·5	+ 2·5	409	446·5	+ 37·5	1840	1885·5	+ 45·5	40 ,, 44
45 ,, 49	373	347·1	− 25·9	369	365·1	− 3·9	1524	1523·4	− 0·6	45 ,, 49
50 ,, 54	267	265·9	− 1·1	275	282·2	+ 7·2	1217	1166·7	− 50·3	50 ,, 54
55 ,, 59	172	180·0	+ 8·0	175	193·0	+ 18·0	778	786·0	+ 8·0	55 ,, 59
60 ,, 64	91	104·9	+ 13·9	126	112·0	− 14·0	464	461·4	− 2·6	60 ,, 64
Totals	3198	3197·7	+ 45·6 − 45·9	3299	3300·6	+ 62·7 − 61·1	14195	14196·1	+ 139·0 − 137·9	Totals
			± 91·5			± 123·8			± 276·9	

TABLE VII.

$O^{[M]}$ *Experience.* 48 *Ages.* *Entry Ages* 17 *to* 64.

Comparison of the results of various Graduations.

Total Deviation of the Graduated Expected Deaths from the Actual Deaths for all the Five Years of the Select Period.

Entry Ages	1 Section Constant Factor	2 Sections Graded		3 Sections Graded		4 Sections Graded		1st Yr. 2 Secs. Others 4 Secs. Graded 4 Yrs.	Entry Ages
		4 Years	5 Years	4 Years	5 Years	4 Years	5 Years		
17 to 19	− 10·1	− 15·6	− 16·4	− 17·4	− 18·3	− 17·4	− 18·3	− 15·1	17 to 19
20 ,, 24	− 26·1	− 59·7	− 64·2	− 71·5	− 78·2	− 72·1	− 78·8	− 63·7	20 ,, 24
25 ,, 29	+ 139·1	+ 94·6	+ 87·1	+ 75·3	+ 67·2	+ 79·0	+ 68·7	+ 81·2	25 ,, 29
30 ,, 34	+ 28·0	+ 3·6	− 1·3	− 9·7	− 13·9	− 0·7	− 5·8	− 5·6	30 ,, 34
35 ,, 39	+ 3·5	+ 2·5	+ 1·5	− 1·3	− 0·6	+ 10·9	+ 10·7	+ 4·3	35 ,, 39
40 ,, 44	+ 20·2	+ 35·8	+ 38·1	+ 40·0	+ 43·7	+ 49·3	+ 54·0	+ 45·5	40 ,, 44
45 ,, 49	− 37·7	− 11·9	− 8·1	− 2·5	+ 2·6	+ 1·1	+ 7·8	− 0·6	45 ,, 49
50 ,, 54	− 86·4	− 57·0	− 51·6	− 44·8	− 38·6	− 49·3	− 42·6	− 50·3	50 ,, 54
55 ,, 59	− 17·0	+ 9·2	+ 14·5	+ 21·1	+ 26·8	+ 9·6	+ 14·6	+ 8·0	55 ,, 59
60 ,, 64	− 13·4	+ 5·8	+ 10·0	+ 15·2	+ 19·1	− 0·1	+ 3·5	− 2·6	60 ,, 64
Totals	+ 190·8 − 190·7	+ 151·5 − 144·2	+ 151·2 − 141·6	+ 151·6 − 147·2	+ 159·4 − 149·6	+ 149·9 − 139·6	+ 159·3 − 145·5	+ 139·0 − 137·9	Totals
	± 381·5	± 295·7	± 292·8	± 298·8	± 309·0	± 289·5	± 304·8	± 276·9	

Table VIII.

(AMP)[M] *Experience.* 44 *Ages.* *Entry Ages* 17 *to* 60.

Four Sections of 11 *Ages each.* 1*st Year not Graded.* *Remaining Years Graded* 4 *Years.*

Entry Ages	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Entry Ages
	1st Year—Not Graded			2nd Year			3rd Year			
17 to 19	10	9·7	− ·3	10	9·5	− ·5	8	9·2	+ 1·2	17 to 19
20 ,, 24	46	43·7	− 2·3	33	38·4	+ 5·4	32	37·5	+ 5·5	20 ,, 24
25 ,, 29	64	66·4	+ 2·4	68	59·1	− 8·9	68	58·9	− 9·1	25 ,, 29
30 ,, 34	73	71·7	− 1·3	71	67·0	− 4·0	64	68·7	+ 4·7	30 ,, 34
35 ,, 39	57	62·6	+ 5·6	53	61·4	+ 8·4	70	63·1	− 6·9	35 ,, 39
40 ,, 44	54	45·7	− 8·3	57	48·6	− 8·4	50	50·2	+ ·2	40 ,, 44
45 ,, 49	24	30·2	+ 6·2	30	36·2	+ 6·2	46	38·3	− 7·7	45 ,, 49
50 ,, 54	21	19·8	− 1·2	26	26·4	+ ·4	29	28·1	− ·9	50 ,, 54
55 ,, 59	10	10·8	+ ·8	15	15·5	+ ·5	15	16·7	+ 1·7	55 ,, 59
60	1	1·9	+ ·9	3	2·9	− ·1	7	3·0	− 4·0	60
Totals	360	362·5	+15·9 −13·4	366	365·0	+20·9 −21·9	389	373·7 Assumed	+13·3 −28·6	Totals
			±29·3			±42·8	375		±41·9	

Entry Ages	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Entry Ages
	4th Year			5th Year			All 5 Years			
17 to 19	3	8·6	+ 5·6	7	8·0	+ 1·0	38	45·0	+ 7·0	17 to 19
20 ,, 24	46	36·1	− 9·9	34	34·3	+ ·3	191	190·0	− 1·0	20 ,, 24
25 ,, 29	50	58·1	+ 8·1	71	56·4	−14·6	321	298·9	−22·1	25 ,, 29
30 ,, 34	68	69·0	+ 1·0	65	67·7	+ 2·7	341	344·1	+ 3·1	30 ,, 34
35 ,, 39	63	63·0	...	59	61·9	+ 2·9	302	312·0	+10·0	35 ,, 39
40 ,, 44	61	50·7	−10·3	49	50·1	+ 1·1	271	245·3	−25·7	40 ,, 44
45 ,, 49	25	39·2	+14·2	33	39·5	+ 6·5	158	183·4	+25·4	45 ,, 49
50 ,, 54	23	29·5	+ 6·5	29	30·2	+ 1·2	128	134·0	+ 6·0	50 ,, 54
55 ,, 59	22	17·3	− 4·7	18	17·0	− 1·0	80	77·3	− 2·7	55 ,, 59
60	3	3·0	...	3	3·0	...	17	13·8	− 3·2	60
Totals	364	374·5 Assumed	+35·4 −24·9	368	368·1 Assumed	+15·7 −15·6	1847	1843·8	+51·5 −54·7	Totals
	376		±60·3	370		±31·3			±106·2	

TABLE IX.

O^M *Experience. Entry Ages 17 to 64. The Kenchington Method.*

1st Year not Graded. Remaining Years Graded 4 Years.

Entry Ages	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Actual Deaths	Graduated Ex. Deaths	Deviation	Entry Ages
	1st Year—Not Graded			2nd Year			3rd Year			
17 to 19	24	20·0	− 4·0	35	30·0	− 5·0	28	31·7	+ 3·7	17 to 19
20 ,, 24	158	162·9	+ 4·9	261	238·8	− 22·2	278	253·5	− 24·5	20 ,, 24
25 ,, 29	305	315·2	+ 10·2	417	464·0	+ 47·0	485	504·4	+ 19·4	25 ,, 29
30 ,, 34	366	337·8	− 28·2	466	492·4	+ 26·4	546	540·3	− 5·7	30 ,, 34
35 ,, 39	277	297·6	+ 20·6	433	425·6	− 7·4	479	473·0	− 6·0	35 ,, 39
40 ,, 44	272	254·1	− 17·9	341	354·1	+ 13·1	392	399·1	+ 7·1	40 ,, 44
45 ,, 49	190	207·5	+ 17·5	290	282·5	− 7·5	302	321·1	+ 19·1	45 ,, 49
50 ,, 54	168	161·5	− 6·5	246	215·1	− 30·9	261	245·5	− 15·5	50 ,, 54
55 ,, 59	109	110·1	+ 1·1	155	143·6	− 11·4	167	165·3	− 1·7	55 ,, 59
60 ,, 64	64	66·4	+ 2·4	88	85·9	− 2·1	95	99·0	+ 4·0	60 ,, 64
Totals	1933	1933·1	+ 56·7 − 56·6	2732	2732·0	+ 86·5 − 86·5	3033	3032·9	+ 53·3 − 53·4	Totals
			±113·3			±173·0			±106·7	
	4th Year			5th Year			All Years—1 to 5			
17 to 19	36	32·0	− 4·0	38	31·8	− 6·2	161	145·5	− 15·5	17 to 19
20 ,, 24	267	257·1	− 9·9	268	255·7	− 12·3	1232	1168·0	− 64·0	20 ,, 24
25 ,, 29	511	519·8	+ 8·8	528	524·3	− 3·7	2246	2327·7	+ 81·7	25 ,, 29
30 ,, 34	550	564·7	+ 14·7	592	575·7	− 16·3	2520	2510·9	− 9·1	30 ,, 34
35 ,, 39	505	499·1	− 5·9	519	514·3	− 4·7	2213	2209·6	− 3·4	35 ,, 39
40 ,, 44	426	426·3	+ 0·3	409	444·2	+ 35·2	1840	1877·8	+ 37·8	40 ,, 44
45 ,, 49	373	345·1	− 27·9	369	363·0	− 6·0	1524	1519·2	− 4·8	45 ,, 49
50 ,, 54	267	265·3	− 1·7	275	281·5	+ 6·5	1217	1168·9	− 48·1	50 ,, 54
55 ,, 59	172	181·3	+ 9·3	175	193·8	+ 18·8	778	794·1	+ 16·1	55 ,, 59
60 ,, 64	91	107·5	+ 16·5	126	114·7	− 11·3	464	473·5	+ 9·5	60 ,, 64
Totals	3198	3198·2	+ 49·6 − 49·4	3299	3299·0	+ 60·5 − 60·5	14195	14195·2	+ 145·1 − 144·9	Totals
			± 99·0			±121·0			±290·0	

ABSTRACT OF THE DISCUSSION.

MR. A. G. PATON said that from the late Sir George Hardy's graduation of the $O^{[M]}$ Tables it would appear that the effect of selection wore off more rapidly at the earlier ages than at the older ages. He (the speaker) did not think that that was quite in accordance with facts. Nor did he think that the effect of selection wore off uniformly throughout the whole of the table. The device employed by Mr. King in this second paper on the construction of select tables enabled him to adhere more closely to the facts than the uniform factor employed in his previous paper and also more closely than Sir George Hardy had in his graduation. Two things in the present paper that appealed to the student were the absence of any great amount of mathematics and the simplicity and rapidity with which the table could be constructed.

Mr. King mentioned incidentally in the paper that no one would think of employing a select mortality table to calculate premiums for temporary assurances. He (the speaker) did not understand what theoretical or practical objection there was to calculating temporary assurance premiums from select tables. The temporary assurance premiums derived by Mr. Elderton from the British Offices' Experience (*J.I.A.*, vol. xxxvii, p. 501) confirmed the fact that the $O^{[NM]}$ Table was quite a satisfactory table to employ in calculating temporary assurance premiums. In addition the $O^{[NM]}$ published Tables gave temporary assurance premiums, both single and annual, for all terms of assurance from one year to ten, and he thought those premiums might be safely used as the basis of rates.

With reference to Mr. King's statement that if the $O^{M(10)}$ Table had been used to graduate his Select Table the effect on the rates of mortality would have been inappreciable, he thought that the rates of mortality arrived at would depend to a certain extent upon the Ultimate Table. In order to test that he regraduated the $O^{[M]}$ data, using the $O^{a/(5)}$ instead of the $O^{M(5)}$ Table and making the rates of mortality constant below age 33. The object was to get a curve which followed a different course from the $O^{M(5)}$. The $O^{a/(5)}$ rates at the younger ages were higher than the $O^{M(5)}$; after age 50 they were lower. Following Mr. King's method exactly he found that the rates of mortality for the younger ages by his (the speaker's) graduation were higher than the rates of mortality obtained by Mr. King, and the rates at the higher ages were lower. In some cases the difference was as much as 5 per-cent of the rate of mortality. It seemed that the tendency was for the graduated results to follow the course of the curve upon which they were based.

MR. A. HENRY had followed with much interest the ingenious process by which Mr. King had made his method somewhat more elastic than it had been in its original form, but he was left with the impression at the end that it was not only the graphic method of graduation which left a considerable amount to the taste and fancy of the operator. As he understood the process, Mr. King divided the data into sixteen sections—being four duration-groups and four

age-groups—and he assumed that what might be called the gradient, that was, the rate at which selection wore off, was constant in each of these, and further that the gradient of the factors themselves was also constant from age to age. That of course was a very suitable device, but although the effect of selection might be different in different age groups it assumed nevertheless that the gradient remained the same from age to age, which might or might not be the case. He could not, therefore, entirely agree with the last speaker that Mr. King's method remedied the apparent defect in Sir George Hardy's graduation. He was doubtful also about making the ratios the basis of graduation. It involved the danger of losing sight of the weight of the data in different parts of the experience.

Mr. King drew attention to a very important point—that the $O^{M(10)}$ Table constructed from the aggregate data showed mortality which averaged something like 1 per-cent higher than the 10-year Ultimate Table which Sir George Hardy constructed from the select data. That was much more important, in his view, than the question which of two or three possible graduations was to be preferred. It raised the old question of the relative merits of policies, lives and amounts. Sir George Hardy, in his lectures on Mortality Tables, laid down the canon that, broadly speaking, amounts should be ruled out for all purposes. Quoting from memory, he believed Sir George said that either the mortality of the big policies was the same as that of the small policies or it was not, that in the former case the rates of mortality would not be altered but the irregularities would be more marked, and that in the other case the data were not homogeneous, and he went on to say that the big policies were not themselves homogeneous, that the experience would differ in different offices, and that therefore on the whole one must base tables on lives. That was probably sound in regard to life office data, but at the same time life offices had to look at the financial results of their policy, and where they ignored those results, as they did by depending on lives, they left something to the arbitrary working out of averages which had not been investigated. In the O^M Table an attempt was made to work on a strict basis of lives throughout, and the point to which Mr. King had drawn attention was evidence that introducing the later contracts gave more weight to the better lives. He thought it was quite conceivable that people who took out a series of policies or annuities might form a more thrifty and possibly better class of life than the single policyholder. It seemed to him therefore that before considering questions of graduation it was necessary to settle questions of principle as to the lines on which the data should be compiled.

Mr. R. D. ANDERSON said that Dr. Sprague had shown how to apply the graphic method to select tables, and Sir George Hardy how to apply Makeham's formula, and now Mr. King had shown how to apply the finite difference method. One of the minor points which had not been dealt with

that evening was the question of the function to be operated upon. He would suggest that the function $\frac{q_{x+t}-q_{(x)+t}}{q_{x+t}-q_{(x+t)}}$ possessed the same advantages as $\frac{q_{(x)+t}}{q_{x+t}}$ and would in addition be found more stable, so as possibly to do away with the necessity of grading. He based that suggestion on the following considerations. The mortality of the first year related to a homogeneous group of lives. He thought it was better to graduate it by itself, as Mr. King had done, and not to grade it like the other years. After that year the mortality of the lives increased as some of them deteriorated and some of the select lives perhaps withdrew, and the constitution of the group changed from all select to mixed, until it gave the ultimate rate of mortality in which the lives were mixed in a proportion which remained fairly stable. That constitution changed gradually as time elapsed, and he thought it was possible to regard the mortality rates of the period of selection as arising out of a group which could be divided roughly into two, one subject to the ultimate rate of mortality and the other consisting of select lives, so that the rate of mortality could be expressed in the form :

$$q_{(x)+t} = aq_{(x+t)} + (1-a)q_{x+t}$$

and

$$a = \frac{q_{x+t}-q_{(x)+t}}{q_{x+t}-q_{(x+t)}}$$

The quantity "a" changed gradually from 1 to 0, and he thought it gave the measure of the persistence of selection, provided that the ultimate table was a true ultimate table. In the case of the use of a table which had not reached ultimate stability, it would not give perhaps a measure of selection but would give a relative measure, and it would be a more constant function than the ratio $q_{[x]+t}$ to q_{x+t}, which involved to some extent the amount of selection. He had investigated the effect of using the proposed function for the second and fifth years of assurance, and he thought the results justified a more extended trial. Taking 10-year age-groups and using Mr. King's $q_{[x+t]}$, he found that the values of a for the second year of assurance were ·638 for 20–29, ·648 for 30–39, ·648 again for 40–49 and ·544 for 50–59, so that, taking that year of assurance, it looked as though the persistence of selection started low and rose to a maximum and then went down again, but the variation was not great. The corresponding results for the fifth year of assurance were : ·315, ·312, ·398 and ·356. There was not much variation there. He had used the average ratios for the whole 40 years of age, namely, for the second year of assurance ·628 and for the fifth year ·337. Multiplying these ratios by the $O^{[M]}$ first year rates and their complements by the $O^{M(5)}$, he obtained the rates in the third column of the following comparative statements :

Second Year of Assurance.

Age at Entry	Attained Age	Rate of Mortality per-cent by	
		Proposed method	Mr. King's method
24	25	·433	·441
29	30	·476	·480
34	35	·541	·543
39	40	·641	·642
44	45	·797	·795
49	50	1·040	1·031
54	55	1·421	1·390
59	60	2·019	1·934

These rates were fairly close to Mr. King's although higher at the older ages. It appeared from the Table of Expected Deaths, given by Mr. King, that his rates were rather under the mortality at the higher ages, so that the difference was on the right side.

Fifth Year of Assurance.

Age at Entry	Attained Age	Rate of Mortality per-cent by	
		Proposed method	Mr. King's method
21	25	·552	·549
26	30	·602	·596
31	35	·678	·672
36	40	·797	·794
41	45	·984	·985
46	50	1·274	1·279
51	55	1·728	1·732
56	60	2·437	2·423

By this method the expected deaths were the same as the actual deaths, as in the case of Mr. King's, and the difficulty that a constant factor understated the mortality at the older ages might be got over.

Mr. C. W. KENCHINGTON did not consider that the modification which he had suggested in connection with Mr. King's former paper would be applicable to all experiences. Where the data were not extensive and the ratio of the actual to the expected ultimate deaths varied from age to age irregularly, it was clear that, if the graduation had to fit with any degree of accuracy, it was necessary to go to greater trouble in curve fitting than merely to adjust a uniform multiplier by a constant.

The device of grading seemed to him to be the special feature of Mr. King's new development of his method. He shared with Mr. Henry some doubt as to the propriety of making the redistribution on the basis of the proportion which a sectional ratio

bore to the total ratio. So far as the British Offices' experience was concerned, since the ratios did not vary greatly according to age in any one year of assurance, and as there was comparatively little variation in the distribution as between the successive years of assurance, it might not matter ; but he thought Mr. King's object of cross-graduation could perhaps have been attained by a method not open to this objection, namely, by operating on the deaths instead of on the ratios of the deaths to the expected. It would merely involve, for the O^M experience, taking the cross cast of the deaths in years of assurance 2 to 5 for each of the four entry age groups, calculating their proportions to the total deaths, and then applying those proportions to redistribute the actual deaths in each of the successive years of assurance, *i.e.*, if it were thought desirable to keep the mortality exactly the same in the graduated experience within each year of assurance. He was not sure that it was necessary to keep the mortality exactly the same as that given by the original data in years of assurance. Mr. King found it necessary to make adjustments in the case of the AMP experience, but he seemed to be satisfied with the original data in the case of the O^M. Certainly the rate of progression from one year of assurance to the next appeared to be fairly satisfactory, but he was inclined to think that it would be possible to secure the object of the grading if it was assumed that the actual deaths over the whole period to be analysed could be redistributed by a rational integral function depending merely on the duration of assurance—as he had suggested in the discussion on Mr. King's former paper. He (the speaker) was then referring to the graduation of the whole period of ten years after selection, and he mentioned a function of the third degree. For grading the four years 2–5 he would use a function of the second degree. Redistributing the deaths by the formula $a+bt+ct^2$, he found for all ages the actual deaths in year of assurance 2 fell short of the redistributed deaths by 3·6 ; in year of assurance 3 the difference was 10·8 negative and in years of assurance 4 and 5 those exact figures were reproduced with opposite signs, showing that the redistribution was of a symmetrical nature. Using the figures redistributed in that way, not only were factors obtained which progressed more smoothly than those derived from the unadjusted data, but the use of a stereotyped form which made the factors the same for each year of assurance was avoided.

Mr. Paton had referred to the desirability of choosing an appropriate ultimate table. He agreed with Mr. Paton's remarks on that subject, because he had found that to obtain the best graduation on simple lines, such as were suggested, it was necessary to use the $O^{M(10)}$ Ultimate Table for the $O^{[M]}$ Experience, and that the $O^{M(5)}$ was really not suitable for the purpose. In that connection and with reference to the difference between mortality tables based on lives and tables based on policies, he would remind the meeting that Mr. F. W. Robertson had constructed a full aggregate mortality table with O^M select data, and had compared the resulting mortality with the mortality of the O^M Aggregate Table. (*T.F.A.*, vol. vii, p. 127).

This gave an exact comparison between policies and lives and showed that throughout the whole table the inclusion of duplicates had a tendency to diminish the rate of mortality. It seemed to him therefore that to obtain a satisfactory junction between the analysed rates of mortality during the period of selection and those of the ultimate table after selection had worn off in the case of a select table based upon policies it was necessary to have an ultimate table also based upon policies.

Mr. R. E. UNDERWOOD referred to Mr. King's remark that "in Australia there is not so much room for gain in selection as at "home, on account of the already low mortality disclosed by the "Ultimate Table." This did not necessarily follow. If the heavy mortality of the one table or country was due to endemic disease, and a great improvement was obtained in the treatment and prevention of that class of disease, the reduction in mortality would not reduce the effect of selection; on the other hand, if it was due to organic disease, than a reduction of organic disease would probably affect the selection considerably. A very much lighter mortality was being experienced in this country at the present time than at the time when the $O^{[M]}$ Table was constructed. The experience of certain offices at the present time probably was not any heavier than that shown by the AMP Table. That might be due to a variety of causes, amongst which were improved sanitary conditions, which possibly would not have any effect on the selection. On the other hand, there were important improvements in surgery and improvements in medicine and preventive medicine which might and probably would have a considerable effect on the selection as exercised by medical men. It was of considerable interest to know what that effect would be, because probably one of the greatest inconveniences of the $O^{[M]}$ Table was the fact that it was found necessary to extend the select period to 10 years. In the next experience it might be found, as in the AMP, that the mortality converged more rapidly and the effect of the selection wore off more quickly, and it might be possible to construct a select table with a period of 5 years or some period less than 10, which would be a great advantage to everyone using it.

Mr. D. C. FRASER remarked that Mr. King's two papers represented two distinct stages in his argument, and it would be convenient to think of the final results now before them as if they were arrived at in two corresponding stages although this involved a departure from the actual arithmetical processes explained in the present paper.

In the first stage five columns of select rates could be obtained by means of five factors precisely as in the first paper. The values of these factors were governed by the period since selection, and did not depend in any way on age.

In the next stage, after setting aside the first column for special treatment, they could obtain Mr. King's results by multiplying the rates in the remaining four columns by the values of a grading function. The grading function was the same for all four columns,

and its values were governed by the age of entry and did not depend in any way on the period since selection. Although this function was not separately stated by Mr. King the columns of "Factors" in Tables V(B) to V(E) were simply the constant factors of the first stage multiplied by the values of the grading function. In the application of the method to the $O^{[M]}$ Experience—to which his subsequent remarks must be understood to refer—the value of this grading function ranged beteween ·98 and 1·02. At the first age of each column, namely, 18 for the 2nd year, 19 for the 3rd year, 20 for the 4th year and 21 for the 5th year, corresponding to age 17 at entry, the value of the grading function was very nearly 1·02. At the 7th age of each column, namely, the age corresponding to age 23 at entry, the value of the grading function was at its minimum ·98. It then gradually increased until at the 36th age in each column, the age corresponding to age 52 at entry, it reached its maximum value, and from that point it diminished to an average value at the final age of each column, being the age corresponding to age at entry 64.

He wished to direct particular attention to the existence of the minimum and maximum values of the function. If they could be accepted as representing facts they suggested that the selection exercised by the Companies had reached its highest point of efficiency at age 23, and was least efficient at age 52. What exactly were the features in the data, which led Mr. King to these minimum and maximum points?

In column (*a*) of the table in § 45 the ratios were given, for the five combined years of the select period, of the actual deaths to the expected deaths by the $O^{M(5)}$ Table. Mr. King remarked in § 46 that for entry ages 17–19, and 20–24 the original data had large ratios, which meant that the rate of mortality had been above the normal; while at entry ages 25–29 the original data showed a small ratio, meaning that the mortality had been unduly light. The ratio at ages 17–19 was as Mr. King pointed out relatively large, but on account of the paucity of data at these ages the deviation from the average value was no more than was to be expected, and therefore no significance could be attached to the high ratio. In the next age group, 20–24, the ratio was 1 per-cent above the average for the column. Here again the deviation was within the expectation. The actual observations, he might point out, were in contradiction to the minimum value shown at age 23 by Mr. King's grading function. At ages 25–29 the ratio was very low, being more than 6 per-cent below the average for the column. The existence of this feature might be due to the fact that the ages of 25–29 were the ages of promotion and of marriage, and that in consequence there was a large volume of highly select business offered to the Companies.

So far it would appear that if there was to be a minimum value in the grading function it should be at age 27 rather than 23.

At the later ages of the Table Mr. King remarked that at entry ages 40–44 and ages 50–54, there were large deviations. But at ages 40–44 the deviation from the average ratio was only 1 per-cent

and was not of any significance. At ages 50–54 mortality had been heavy in the select period and the ratio was between 8 and 9 per-cent in excess of the average.

They were left with the facts that at entry ages 25–29 there had been an exceptionally light ratio of select mortality and at ages 50–54 an exceptionally heavy ratio. When the ratios were plotted out for the whole series of entry ages, those facts stood out very prominently, there being a very low dip at ages 25–29 and a very high peak at ages 50–54. These were the actual facts that stood behind the occurrence in Mr. King's grading function of a minimum at age 23, and a maximum at age 52.

Apart from the question whether Mr. King had found the most appropriate expression for these facts there was the question whether the facts were of such a character that expression ought to be given to them.

As Sir George Hardy had pointed out the object of a graduation was something more than merely to remove the accidental irregularities which are inherent in all statistical data. The uses which the graduated Tables were to subserve had to be borne in mind.

The low dip at 25–29 was a very strongly marked and isolated feature. The reason for it could be understood and they might hope that it would prove to be a permanent feature in their experience. In his former paper Mr. King had pointed out (*J.I.A.*, vol. lii, p. 297) that according to the original data, the probability of death within five years after entry was lower for entry ages 25–29 than for entry ages 20–24. But was this not just the kind of fact they must discard? The facts might suggest descending rates of mortality at a certain range of ages, but they had only to consider what a revolution would be made in their practice if they had to deal with descending instead of ascending rates of mortality to see how necessary it was to eliminate such a feature. He suggested therefore that while the light mortality at ages 25–29 might be accepted as a fact, it should be boldly cut out, in view of the greater importance of securing a proper and consistent progression in their mortality rates.

The high ratio at 50–54 could not be readily accounted for. It might well be that it was an accident of the experience, which need not necessarily be expected to recur, and he suggested that as in the case of the earlier dip, the peak at these ages should be cut out. This was what had been done in the select experience, and on examination it would appear that compensation had been secured by increasing the deaths in the immediately succeeding portion of the $O^{M(5)}$ Table, which supplied a further argument against distributing the excess of deaths over the neighbouring ages in the select period.

These, however, were points of detail. Returning to the general principle of the grading function he would point out that the employment of a uniform grading function for the 2nd, 3rd, 4th and 5th years of the select period prevented a smooth junction with the

rates of the $O^{M(5)}$ Table. The difference between the values of $q_{[x]+4}$ and q_{x+4} was about 25 per-cent of the value of $q_{[x]+4}$ at the extremes of the table, but fell as low as 20 per-cent at intermediate ages, and he could not but feel that these fluctuations were greater than they should expect to find in a scientifically constructed table.

If they considered the rates of mortality in the select period which appeared on the same horizontal lines, that is, if they considered them with reference to ages attained instead of with reference to the period since selection, they would find that at every age the benefit of selection ran off by the same stages after the first year until the end of the 5th year. This was clearly indicated by Mr. King in § 35, where he showed that at every age the rate of mortality was 13·6 per-cent greater in the 3rd year than in the 2nd, 20·7 per-cent greater in the 4th year than in the 2nd, and 24·5 per-cent greater in the 5th year than in the 2nd. This was a result of using a uniform grading function; in other words, the idea of a uniform grading function and the idea of uniform benefit of selection were bound up together.

If the benefit of selection were uniform from the end of the 1st year to the end of the 5th year he felt personally that he could not resist the inference that the uniformity must extend further, and must continue as they made the transition from the values of the 5th year to the values of the Ultimate Table. But as he had pointed out the junction between the values of the 5th year and the values of the Ultimate Table was not by any means uniform in its character, and he was driven to the conclusion that if the grading function were uniform a smooth junction with the ultimate values could not be secured unless its values were equal to unity throughout, so that they were brought back to the process of Mr. King's first paper; with possibly a modification for the first year of the select period.

It might be possible to secure a smooth junction with the values of the Ultimate Table by using a varying grading function, the values of which should approach nearer to unity as the period since selection increased, reaching unity at the point of junction with the select period. He suspected, however, that if they wished to elaborate on the simplicity and beauty of the method brought before them in Mr. King's first paper they would find themselves returning to the methods of Sir George Hardy, who took advantage of the fact that the $O^{M(5)}$ Table followed Makeham's Law and dealt separately during the select period with the A and B terms in the expression for the force of mortality.

He had confined himself to a single aspect of Mr. King's paper and he wished to say in conclusion that Mr. King had rendered a most valuable service in his two papers by re-opening the question of the graduation of select tables. He had cleared and simplified the approach to a difficult subject and he felt sure that further consideration of the subject, whatever their final conclusions might be, would take its starting point from the methods laid before them by Mr. King.

The PRESIDENT said that Mr. King had given them a most

excellent and stimulating paper. He had taken up the subject at the point at which it was left by the discussion of a year ago and had dealt faithfully and illuminatingly with each one of the points of importance then raised, and in so doing, he had given Mr. Kenchington and others an opportunity of further developing the ideas that they then threw out. He was sure that in no respect would the reception of the paper have given Mr. King more satisfaction than in the fact that it had produced various expressions of opinion from the younger members of the Institute. The development of select tables represented a lengthening history beginning, so far as definite effort was concerned, with A. J. Finlaison's "Government Annuity Experience", which, if he remembered rightly, graduated downwards but not across. This was followed by Dr. Sprague's scientific researches into the subject, and later by the great work of Sir George Hardy, in which, after laying down certain general principles as the basis upon which his work should proceed, Sir George was led to adopt mathematical expedients in modification of the general scheme to meet the difficulties which had to be overcome. He dealt with his problems in a way that such a master mind would do—that is, by the application of plans devised to meet the particular difficulties encountered; it was not to be assumed that he would necessarily have confined himself to the same procedure in dealing with any subsequent production of the same kind. This seemed to him (the President) an important point to be borne in mind in the study of Sir George Hardy's methods. From that beginning, by the inevitable course of evolution, a simplified form appropriate to the treatment of a wide range of cases had now come, and, so far as the daily work of the actuary was concerned, he thought Mr. King's latest paper had provided them, as did the paper he read a year ago, with a most excellent piece of working machinery.

Mr. KING, in reply, said the discussion had taken the line he most appreciated. He had listened with especial pleasure to Mr. Anderson and Mr. Kenchington, as their remarks gave ground for the hope that further papers would be forthcoming giving other methods of constructing select tables which might be better than his own. Even if they should not prove better in practice, it was desirable to have various methods. His difficulty throughout had been to join all the five years together. He hoped Mr. Anderson's or Mr. Kenchington's method would enable that to be done, and would provide good tables where all the columns were co-ordinated, which was what was really required. Any method of this kind had to be worked out fully to see if it produced consistent results throughout the whole of the select period.

He welcomed Mr Henry's remark as to discretion being required in using his method. That was wanted in any method of graduating tables. He did not expect that a mere mechanical application of his method, or of any other, would bring out the best results in any particular case. Each case must be studied, and discretion used in applying the method in a proper way.

Perhaps he had been a little too emphatic in saying without fuller explanations that the Ultimate Table had not much effect on the rates of mortality but affected only the factors, but he thought that Mr. Paton had omitted to notice that the first step in the short method of constructing select tables was to prepare an ultimate table. It was only when the ultimate table was reasonably based upon the same experience as the select that it could be said there was not much effect produced by a change in the ultimate table. This was so in the case of the $O^{M(5)}$ and the $O^{M(10)}$. If Hardy's edition of the $O^{M(10)}$ were used instead of the $O^{M(5)}$ the factors would be changed, but he did not think the values of q would be affected appreciably. Even taking the $O^{M(5)}$ Table for the Ultimate in the AMP select experience, there would not be very serious differences in the resulting rates of mortality, but the AMP Ultimate Table could not be used for the British Offices' Experience. That would not give proper factors, one reason being that the factors throughout a great part of the table would come out greater than unity, which was, of course, not legitimate. When he said that the effect of changing the ultimate table did not affect to any appreciable extent the select rates of mortality, he implied that it was necessary to have a table really based upon the same or similar data. Moreover, if the Ultimate Table be changed, the application of the method of constructing the select columns might also have to be varied. It might be necessary, for instance, to use a different range of entry ages, or a different number of sections in the columns, or even a different number of columns to be graded. In fact, the experience would have to be re-examined in the new light. With regard to term insurances, the difficulty was that the mortality was very irregular, and the term insurances were apt to be of a very varying magnitude, some of them very large. Possibly, with the greater knowledge now possessed, it might be possible to graduate the term insurance data of the British Offices in select form, and in a way that would bring out its real meaning.

The most important question which had been raised was that of duplicates, although it was only incidental to the present paper. The British Offices' Experience Committee, which was a very strong one, was unanimous that it was desirable to exclude duplicates, and the explanation given in "Principles and Methods" showed with what care this had been done. Mr. Kenchington had said that as policies were used in the select columns, it was necessary to use them in the ultimate column. But the select columns were not based on policies, but on lives. Nowhere in the select tables did any life appear more than once, all duplicates being excluded. The same principle should apply to the ultimate table; at no point of the ultimate table should a life appear more than once. It spoilt the table to bring in duplicates. Duplicates were not brought into Hardy's $O^{M(5)}$ Table, and he had not realized until the present paper was well advanced that Hardy's $O^{M(10)}$ Table included duplicates. In anticipation of the discussion, he had tried to set out clearly his reasons for thinking that duplicates should not be

included in the experience. They were as follows: Suppose that four lives, A, B, C and D, all assured at age x in an experience where no one had effected duplicate policies. It would be generally admitted that the best available measure of their probabilities of life was given by a table constructed in the select and ultimate form. Suppose now that the experience was altered by B, C and D having taken out further policies at ages y, z, etc., later than age x, while A remained content with his one policy. Those duplicate policies of B, C and D would be rightly included in the select columns, but that did not affect A, as the duplicates did not appear in the table for entry age x in the select columns. If, however, they were retained in the ultimate column formed from lives assured for ten years or more, they might affect A, in that they might change the measure of his probability of life and cause it to depart from what had already been admitted to be the best available measure. Surely that should not be. Moreover, even as regards B, C and D, their probabilities of life were not really affected by their taking out the duplicates. That they were accepted proved only that they still remained in the category of unimpaired lives when they effected their duplicates. Therefore their probabilities of life should still be measured by an ultimate table excluding duplicates. In the British Offices' Experience, the effects of selection were perceptible even after the expiry of ten years. That fact, however, strengthened the argument against the inclusion of duplicates in the $O^{M(10)}$ Table. In the case of the first policies of A, B, C and D, effected at, say, age 25, the effects of selection would have ceased after twenty years, by, say, age 45; whereas with new entrants at, say, age 35, who passed into the ultimate table at age 45, the effects of selection would not by then be exhausted; and for that cause the ultimate mortality would probably be underestimated at attained age 45, and for some years thereafter; and to bring in also the duplicate policies of B, C and D had a tendency to increase that error. It seemed to him, therefore, that there was no justification for introducing deliberately a further error in the same direction into the ultimate table, even although that table was already imperfect.

From those considerations the conclusions were reached that the first edition of the $O^{M(10)}$ Table, from which duplicates had been excluded, already gave a value a trifle too small for the measure of mortality of lives who entered at the younger ages, and that the second edition, which included duplicates, made matters worse in that respect. The counsel of perfection would be to prepare select tables for each entry age, and to abolish altogether the ultimate table; but that course in practice would be impossible, because of the immense mass of tabular matter which would be necessary. The main argument for retaining duplicates in the ultimate table appeared to be that thereby it was said to become more easy to join on satisfactorily the select portion of the table to the ultimate; but that argument to his mind was not legitimate. It was the business of the actuary to adapt his formulas and methods to the data, and

not to manipulate the data to suit his formulas and methods. In the case of the British Offices' Experience, it had been shown that it was not possible to have a Makeham $O^{M(10)}$ Table from which duplicates were excluded; and therefore the question was brought down to the one very narrow issue, namely, to what extent accuracy could be properly sacrificed in order to secure the practical advantage of having a select and ultimate table which followed Makeham's Law throughout. In the case of any new experience of assured lives or annuitants, he hoped that it would be decided—even should it be resolved to include duplicates in the ultimate table—to put the data in such a form that tables could be prepared excluding duplicates. It was a question of opinion on which people might properly differ, and for that very reason he did not think the facts should be so recorded that those who thought duplicates should be excluded could not use the data to bring out what they thought the best tables.

It seemed to him also that the argument is fallacious that financial results can best be measured by including duplicates in a table, because so much depends on the proportionate number of duplicates; and the measure is disturbed by increases or decreases in the proportion. A table based on lives is the only thing really definite, and such a table should be constructed. Then if duplicates are to be considered, a table could be constructed from them alone, omitting the first policy on a life; and, by comparing this second table with that based on lives only, suitable allowance could be made for any particular proportion of duplicates in an experience.

[We have received the following communications from Mr. King and Mr. R. D. Anderson.—Eds. *J.I.A.*]:

Mr. King writes:

Mainly on account of inability to hear all that was said at the Institute in the discussion which followed the reading of the paper, I could not reply on the spot to some of the points raised, and I have now to thank the Editors of the *Journal* for their courtesy in allowing me to supplement the remarks I then made.

I should like to add a few words to my reply to Mr. Henry. He said of the "gradients" that they are the rates at which selection wears off; that they are constant in each section of each graded column, and that the gradient of the factors themselves is constant from age to age. I venture, however, to think that there is a misunderstanding here, and that Mr. Henry and I are speaking of different things, because his description does not apply to the "gradients" of the paper (see par. 29) from which the term "grading" has been derived. The gradients of the curves representing the successive columns of factors are measured by the first differential coefficients of the factors, taking the entry age as the

independent variable. A short demonstration will make their nature clear.

For brevity in notation let u_x represent for the moment the factor for entry age x of the column for the first graded year, and w_x the factor, again for entry age x, for the column t years later after selection. It has been shown in the paper (see par. 35) that $w_x = cu_x$ where c is a constant. The gradient of the curve representing the column of u is $\frac{du_x}{dx}$; and the gradient of the curve representing the column of w is $\frac{dw_x}{dx}$, which is equal to $c\frac{du_x}{dx}$. Thus the gradients are not identical, but, in the words of par. 29, they bear at all points a constant relationship to each other.

Also, the gradient of the factors is not usually constant from age to age. Let $w_{[x]+n}$ represent the factor for entry age x of a column n years after selection; then

$$q'_{[x]+n} = w_{[x]+n} \times q_{x+n}$$

Whence
$$w_{[x]+n} = \frac{q'_{[x]+n}}{q_{x+n}}$$

Taking now the differential coefficients it is seen that the gradient is a function of the first differential coefficient of the ratio of the select rate of mortality to the ultimate rate, and that it is not constant from age to age except in the special cases of (1) a factor changing by a constant first difference, when the gradient is constant, and equal to the difference; or (2) of a factor uniform for a whole column, when the gradient has the value zero at all ages.

It is with great reluctance that I proceed to comment on Mr. Fraser's contribution to the discussion, because it is very distasteful to have to point out what seem to me to be misconceptions on matters vital to the paper. Had any one of lesser weight made the speech, I should pass it over almost in silence, but with one like Mr. Fraser, so justly and universally esteemed in the profession, judgment cannot be allowed to go by default.

Mr. Fraser's opening sentences are not easy to understand. He speaks of what *could* be done, but that is a matter of opinion on which we differ. As to his summary of what actually is done, I need say only that to my mind it is not accurate, because we do not multiply the "constant factors" of the former paper by a "grading function" in order to form the new factors, but we operate on ratios of actual deaths to ultimate expected deaths in the sections, not used in the former paper, into which the columns are divided, and from which the factors are subsequently derived. The factors themselves could not be treated in this way.

There is nothing in the paper that can reasonably be called a "grading function", but there is a comparatively simple process, purely arithmetical, by which the graded columns in the select period are brought into harmony with each other by being brought

into conformity with the general law derived from the experience as a whole. By that process such irregularities are eliminated as Mr. Fraser himself says should be "boldly cut out." This is set forth with some detail in pars. 24 to 26 of the paper, and I venture to submit that these, and especially par. 25, prove that not only is it legitimate, but that it is the proper course to be followed, to provide that the several graded columns shall comply with the general law. It is a pity that this important point was not taken up more fully at the Institute. It was hardly even mentioned.

In dealing with my graduation of the British Offices' Experience, Mr. Fraser calls "particular attention to the existence of minimum and maximum values of the function", and analyses minutely the columns to find them; and, looking only at the columns of factors, he picks out the ages at which they occur, while at the same time the context seems to show that they are treated as belonging also to the columns of the rates of mortality. In reality they occur only in the columns of factors; and the form of the factors is determined by the nature of the ultimate table. Now, the ultimate table employed is Hardy's $O^{M(5)}$, and the Makeham law involved in it has the effect of overstating the rates of mortality at the younger ages to about age 25. The consequence is that the factors at the early entry ages necessarily come out small, in order to produce the true rates of mortality in the select columns. That accounts for the minimum at entry age 23 objected to by Mr. Fraser. There are not any maximum or minimum points in the rates of mortality, but these increase constantly at an accelerating pace from the youngest age to the oldest.

Mr. Fraser discusses at some length col. (*a*) in the table in par. 45 of the paper, the column which gives ratios for the *original data*; and he shows that there is a "very low dip" at entry ages 25 to 29, and a "very high peak" at entry ages 50 to 54; and he asks whether these "facts were of such a character that expression ought to be given to them." He then goes on to enquire what Sir George Hardy would have done under like circumstances, and he concludes by saying that both these irregularities should be "boldly cut out." By stopping short at this point he implies distinctly that I have retained them, but had he proceeded a little further, and called attention to cols. (*b*) and (*c*) of the table, which give the *graded ratios*, in contradistinction to the ratios for the original data in col. (*a*) of which alone he speaks, most likely he would have said that by the grading process the irregularities have been entirely removed; and he might even have offered congratulations on the success of the operation.

Towards the end of Mr. Fraser's speech there is a case of confusion between rates of mortality and factors. He says that in par. 35 I "showed that at every age the rate of mortality was 13·6 per-cent greater in the third year than in the second", and so on; but if he will glance at the paragraph again he will see that that is a mistake. It is the factors as for *entry* ages that bear this relationship to each other, and not the rates of mortality; and,

seeing that the factors are multiplied into the ultimate rates of mortality at *attained* ages, Mr. Fraser will find that the select rates of mortality are not so related to each other.

If the new method is disfigured by all the shortcomings attributed to it by Mr. Fraser, then the results which it produces must be far from satisfactory; but what are the facts?

Mr. Fraser takes as his standard Hardy's construction of the British Offices' Tables, and so do I. In the words of the former paper, Hardy has given us "magnificent select tables", and I need hardly say that in entering on the present enquiry I had no thought of trying to improve upon them. By a master-stroke of genius Hardy has produced select tables of ten columns, following Makeham's law throughout, which are practically perfect, the only blemish, if indeed it be a blemish, being, that after about entry age 45 the rates of mortality in the first five select columns are slightly understated. This is a trifling price to pay for the advantages of Makeham's law. It causes an underestimate of roughly five farthings per pound sterling of the annual premium for a whole-life policy.

Comparing now the results of my own efforts with this perfection, right through all the five columns with which I dealt, my values of q to about entry age 45 are almost identical with Hardy's, never at any age in any of the first four columns differing by more than 2 in the 4th decimal place, and generally by not more than unity; while in the 5th column the difference never exceeds 3 in the 4th decimal place. The differences would not be less between graduations of a table by any two good summation formulas. After entry age 45 the same harmony continues between Hardy's tables and mine, but my values of q are slightly higher than his, and at these ages my tables adhere more closely to the original facts, but they do not follow Makeham's law. Hardy's method of construction and mine are wide as the poles asunder, and it is extraordinary, and very gratifying to me, that they should produce results so nearly identical.

In conclusion I thank the President sincerely for the more than kind and generous expressions which he used towards myself.

Mr. R. D. Anderson writes as follows:

Referring to my remarks on Mr. King's paper, I have now calculated the rates of mortality for the second, third, fourth and fifth years of assurance by the method I proposed, and enclose the results.

In order to restrict the differences between the two graduations as far as possible to differences inherent in the two methods, I have followed Mr. King in the range of ages covered and I have used his rates of mortality for the first year of assurance, for which year the rates must be obtained by an independent graduation if the proposed method be used.

Table I shows the rates of mortality per-cent for the same ages as Mr. King's Table V, the rates for the first year being taken from Mr. King's Table V(A). It will be seen that throughout the main part of the table there is no appreciable difference between the rates by the two methods. The principal divergences are at the ends of the table, where generally the rates by the proposed method are lower than Mr. King's at young ages and higher at old ages.

The rates of mortality were calculated from the formula

$$q_{[x]+t} = aq_{[x+t]} + (1-a)q_{x+t}$$

where a has the following values:

Year of Assurance	a
2	·62806
3	·47039
4	·38807
5	·34497

The values of a were obtained by calculating the expected deaths for each year of assurance for the ages included in the tables according to (1) the rates of mortality for the first year and (2) the $O^{M(5)}$ rates.

The difference between the expected deaths by these two bases divided into the difference between the $O^{M(5)}$ expected deaths and the actual deaths gives the factors above-mentioned. Thus the expected deaths for the second year of assurance for attained ages 18–65 inclusive were 1871·7 by Mr. King's rates for the first year. The $O^{M(5)}$ expected deaths were 4184·7 and the difference is 2313. The actual deaths were 2732 and the difference between these and the $O^{M(5)}$ expected deaths 1452·7. This number divided by 2313 gives the factor ·62806.

In order to calculate the expected deaths for ages 64–68 inclusive, the rates of mortality for the first year of assurance were required for these ages and Mr. King's Table of q_x was extended to 68, the following being the rates used:

Age	q_x
65	·02207
66	·02394
67	·02599
68	·02825

Table II shows the actual and expected deaths for comparison with Mr. King's Table VI. The results are again not very different

from Mr. King's and taken with Table I I think show that substantially the same results can be obtained by a somewhat simpler process.

The use of a constant value of the factor a for all ages in the same year of assurance assumes that selection wears off at the same rate at each age. In order to examine how far this is the case I have calculated the value of a for the same age groups as those used for the comparison of the actual and expected deaths (Table III). The deviations in the deaths naturally correspond to the deviations in the value of a for the group from its value for the whole year of assurance.

The value of a is, of course, affected by accidental deviations in the deaths and in order to form an opinion whether the deviations of a for the smaller age groups from the value of a for the whole year of assurance are more than is legitimate, it is necessary to have some idea of the extent to which the values may be affected. I have therefore calculated approximately the Mean Expected Deviations in a by taking the Mean Expected Deviations in the deaths as equal to $\cdot 8\sqrt{\text{expected deaths}}$. The resulting Mean Expected Deviations are compared with the Actual Deviations in Table IV.

On the whole the use of a constant factor does not appear quite justifiable theoretically on account of the divergences at ages 17–24, but the question is affected somewhat by the use of the $O^{M(5)}$ as the ultimate table, and for a practical graduation if it is desired to produce a smooth table quickly for a special purpose, I think the divergences are not so great as to condemn the method.

TABLE I.

British Offices Select Table. Rates of Mortality per-cent.

The rates for the 1st year are taken from Mr. King's Table V(A). The rates after 5 years are the $O^{M(5)}$ Rates and the rates for the remaining years are interpolated between the rates for the 1st year and the $O^{M(5)}$ Rates by the formula $q_{[x]+t} = aq_{[x+t]} + (1-a)q_{x+t}$.

Age at Entry	1st Year	2nd Year	3rd Year	4th Year	5th Year	After 5 Years
(x)	$100q_{[x]}$	$100q_{[x]+1}$ $(=62{\cdot}806q_{[x+1]} + 37{\cdot}194q_{x+1})$	$100q_{[x]+2}$ $(=47{\cdot}039q_{[x+2]} + 52{\cdot}961q_{x+2})$	$100q_{[x]+3}$ $(=38{\cdot}807q_{[x+3]} + 61{\cdot}193q_{x+3})$	$100q_{[x]+4}$ $(=34{\cdot}497q_{[x+4]} + 65{\cdot}503q_{x+4})$	$100q_{x+5}$
17	·247	·396	·461	·499	·522	·665
18	·250	·399	·467	·505	·528	·672
19	·253	·405	·472	·511	·534	·680
20	·258	·410	·478	·516	·541	·689
21	·262	·415	·483	·523	·549	·698
22	·267	·420	·490	·531	·557	·709
23	·271	·426	·498	·539	·566	·721
24	·276	·433	·505	·548	·576	·732
25	·282	·441	·514	·558	·586	·747
26	·288	·448	·523	·567	·598	·762
27	·294	·457	·533	·580	·611	·777
28	·301	·466	·544	·592	·624	·796
29	·308	·476	·556	·605	·640	·816
30	·316	·488	·568	·621	·657	·837
31	·325	·498	·583	·637	·675	·860
32	·333	·512	·599	·654	·694	·886
33	·344	·526	·615	·673	·716	·915
34	·355	·541	·634	·695	·740	·945
35	·366	·558	·654	·718	·765	·978
36	·379	·576	·677	·743	·793	1·015
37	·393	·597	·700	·770	·824	1·056
38	·408	·618	·726	·800	·858	1·099
39	·424	·641	·754	·834	·894	1·146
40	·442	·667	·786	·869	·934	1·200
41	·461	·696	·820	·907	·979	1·256
42	·483	·727	·857	·951	1·026	1·320
43	·506	·760	·898	·997	1·080	1·388
44	·531	·797	·942	1·050	1·137	1·463
45	·559	·837	·992	1·105	1·199	1·545
46	·589	·882	1·045	1·166	1·268	1·634
47	·623	·930	1·103	1·233	1·342	1·731
48	·659	·983	1·167	1·306	1·424	1·839
49	·698	1·041	1·237	1·386	1·515	1·956
50	·742	1·103	1·312	1·474	1·613	2·083
51	·789	1·172	1·397	1·570	1·720	2·222
52	·841	1·249	1·488	1·674	1·837	2·375
53	·899	1·331	1·588	1·789	1·966	2·541
54	·961	1·422	1·697	1·914	2·105	2·722
55	1·030	1·520	1·817	2·051	2·258	2·921
56	1·105	1·629	1·947	2·200	2·426	3·138
57	1·188	1·748	2·092	2·364	2·609	3·373
58	1·278	1·877	2·246	2·543	2·808	3·632
59	1·377	2·020	2·417	2·737	3·027	3·912
60	1·486	2·175	2·603	2·951	3·264	4·221
61	1·605	2·344	2·807	3·183	3·526	4·557
62	1·735	2·530	3·029	3·439	3·809	4·918
63	1·878	2·733	3·298	3·716	4·118	5·317
64	2·034	2·956	3·538	4·018	4·457	5·748

Table II.

$O^{[M]}$ *Experience. Ages at Entry* 17 *to* 64.

Age at Entry	1st Year			2nd Year			3rd Year		
	Actual Deaths	Graduated Expected Deaths	Deviation	Actual Deaths	Graduated Expected Deaths	Deviation	Actual Deaths	Graduated Expected Deaths	Deviation
17–19	24	19·9	− 4·1	35	28·9	− 6·1	28	31·3	+ 3·3
20–24	158	164·6	+ 6·6	261	233·2	−27·8	278	251·3	−26·7
25–29	305	320·8	+15·8	417	457·4	+40·4	485	502·1	+17·1
30–34	366	343·9	−22·1	466	489·1	+23·1	546	539·4	− 6·6
35–39	277	300·8	+23·8	433	425·9	− 7·1	479	473·3	− 5·7
40–44	272	253·5	−18·5	341	356·4	+15·4	392	399·8	+ 7·8
45–49	190	204·2	+14·2	290	286·0	− 4·0	302	322·0	+20·0
50–54	168	157·2	−10·8	246	218·9	−27·1	261	246·8	−14·2
55–59	109	106·6	− 2·4	155	147·2	− 7·8	167	166·8	− 0·2
60–64	64	64·3	+ 0·3	88	88·7	− 0·7	95	100·2	+ 5·2
Totals	1,933	1,935·8	+60·7 −57·9	2,732	2,731·7	+79·6 −79·9	3,033	3,033·0	+53·4 −53·4

Age at Entry	4th Year			5th Year			All Five Years		
17–19	36	31·8	− 4·2	38	31·7	− 6·3	161	143·6	− 17·4
20–24	267	256·6	−10·4	268	256·0	−12·0	1,232	1,161·7	− 70·3
25–29	511	519·6	+ 8·6	528	525·6	− 2·4	2,246	2,325·5	+ 79·5
30–34	550	565·1	+15·1	592	577·0	−15·0	2,520	2,514·5	− 5·5
35–39	505	499·5	− 5·5	519	514·7	− 4·3	2,213	2,214·2	+ 1·2
40–44	426	426·2	+ 0·2	409	444·0	+35·0	1,840	1,879·9	+ 39·9
45–49	373	344·9	−28·1	369	361·9	− 7·1	1,524	1,519·0	− 5·0
50–54	267	265·2	− 1·8	275	280·6	+ 6·5	1,217	1,168·7	− 48·3
55–59	172	181·4	+ 9·4	175	193·1	+18·1	778	795·1	+ 17·1
60–64	91	107·8	+16·8	126	114·6	−11·4	464	475·6	+ 11·6
Totals	3,198	3,198·1	+50·1 −50·0	3,299	3,299·2	+58·7 −58·5	14,195	14,197·8	+149·3 −146·5

Table III.

British Offices Select Tables.

Values of the function $a = \dfrac{O^{M(5)}\ \textit{Expected Deaths} - \textit{Actual Deaths}}{O^{M(5)}\ \textit{Expected Deaths} - O^{[M]}\ \textit{Expected Deaths}}$

Age at Entry	2nd Year	3rd Year	4th Year	5th Year	Years 2–5 inclusive
17–19	·4155	·5970	·2209	·0837	·3382
20–24	·5029	·3409	·3338	·2813	·3697
25–29	·7245	·5131	·4103	·3386	·5027
30–34	·6821	·4549	·4252	·3072	·4713
35–39	·6086	·4540	·3718	·3326	·4436
40–44	·6810	·4979	·3893	·4652	·5089
45–49	·6096	·5607	·2629	·3140	·4361
50–54	·4637	·3840	·3772	·3770	·4003
55–59	·5544	·4665	·4741	·5053	·4999
60–64	·6402	·5522	·6551	.1690	·5006
All Ages	·6281	·4704	·3881	·3450	·4600

TABLE IV.

British Offices Select Tables.

Comparison of the Approximate Mean Expected Deviations of the function a with the Actual Deviations from the value for all ages combined.

Age at Entry	2ND YEAR		3RD YEAR		4TH YEAR		5TH YEAR		YEARS 2–5 INCLUSIVE	
	Approximate Mean Expected Deviation	Actual Deviation	Approximate Mean Expected Deviation	Actual Deviation	Approximate Mean Expected Deviation	Actual Deviation	Approximate Mean Expected Deviation	Actual Deviation	Approximate Mean Expected Deviation	Actual Deviation
17–19	·1514	+·2126	·1703	−·1266	·1490	+·1672	·1883	+·2613	·0860	+·1218
20–24	·0551	+·1252	·0615	+·1295	·0653	+·0543	·0679	+·0637	·0311	+·0903
25–29	·0408	−·0964	·0449	−·0427	·0473	−·0222	·0488	+·0064	·0227	−·0427
30–34	·0409	−·0540	·0447	+·0155	·0465	−·0371	·0481	+·0378	·0225	−·0113
35–39	·0456	+·0195	·0494	+·0164	·0514	+·0163	·0525	−·0124	·0248	+·0164
40–44	·0519	−·0529	·0557	−·0275	·0575	−·0012	·0584	−·1202	·0279	−·0489
45–49	·0602	+·0185	·0643	−·0903	·0661	+·1252	·0668	+·0310	·0322	+·0239
50–54	·0717	+·1644	·0761	+·0864	·0780	+·0109	·0787	−·0320	·0381	+·0597
55–59	·0911	+·0737	·0961	+·0039	·0977	−·0860	·0979	−·1603	·0497	−·0399
60–64	·1220	−·0121	·1286	−·0818	·1315	+·2670	·1315	+·1760	·0643	−·0406

Results of an Investigation into the effect of different valuation bases upon Surplus. By C. H. MALTBY, F.I.A., *of the Gresham Life Assurance Society, Limited.*

[Submitted to the Institute, 27 February 1922.]

Introduction. MANY investigations have been made in the past to ascertain the surplus, or the bonuses, which would arise with a given valuation basis, from different assumptions as to the actual experience and premiums of an office; but there is little on record regarding similar efforts which may have been made to see how the surplus year after year would be varied by the introduction of different valuation bases, the other two factors remaining constant.

There are very many difficulties in making such an investigation, since the results necessarily depend upon the distribution of the business, and, generally speaking, will differ with each different distribution; but we can trace what happens in individual cases and see how much apparent surplus is produced year by year. A study of these results should then enable us to form a more accurate opinion of the way in which valuation bases and methods affect the surplus disclosed.

Method adopted. In the present investigation I have accordingly adopted the method of tracing the surplus* earned during a policy year at quinquennial intervals under a policy for £100. Only the surplus from mortality, interest and loading is dealt with; the absence of standard data making it almost impossible to include the so-called "minor" sources of profit such as surrenders. The question of surplus arising from alterations in the value of the assets is likewise left untouched.

The formulæ used in the investigation are:

For "Non-profit" Policies.

$$B^1_n=\{[V_{n-1}+\pi]\,.\,\Delta i+[1-V_n]\,.\,\Delta q_{n-1}+[P-\pi][1+i']\}\,.\,p'^{-1}_{n-1} \quad . \quad (1)$$

For "With profits" Policies.

$$B^2_n=[V_{n-1}+\pi]\,.\,\Delta i+[1-V_n]\,.\,\Delta q_{n-1}+[P-\pi][1+i']$$
$$+S_{n-1}\,.\,\{A_{n-1}\,.\,\Delta i+[1-A_n]\,.\,\Delta q_{n-1}\}$$
$$=p'_{n-1}\,.\,B^1_n+S_{n-1}\,.\,b^2_n \quad . \quad (2)$$

* Throughout, the term "surplus" is used irrespective of whether the particular amount in question is in fact positive or negative; the word "profit" being used as a convenient synonym when the surplus is positive.

The numerical values of P in the two formulæ differ, since in (2) the bonus loading is included. The rationale of these formulæ and a note of any special symbols and terms used in the paper, are given in the Appendix.

The experience rates of mortality and interest assumed are of secondary importance, since these have but little influence upon the alteration in the surplus produced by a change in the valuation bases; and the $O^{[M]}$ Table with interest at 4 per-cent was accordingly used as a convenient approximation to the rates probably experienced at the present day. It was also thought desirable to eliminate any real profit. This can only arise from the difference between the rates of mortality, expenses, and interest experienced and those assumed by the office premiums, and in order that the surplus shown should be entirely due to the valuation bases, the values of P (office premiums less the actual expenses) used were $O^{[M]}$ 4 per-cent net premiums.

This course appeared preferable to the alternative of making a selection from amongst the large number of combinations of various scales of office premiums and expenditure which could be justified. The use of any other scale of P (P′ say) only involves in practice the addition of $[P'-P]$ $[1\cdot04]$ to the values given, and in Table II the effect of using $O^{[M]}$ $3\frac{1}{2}$ per-cent values for P has been shown. The use of level values of P throughout the duration of a policy involves the assumption of level expenses; that is that each policy contributes a constant annual amount to a central fund, out of which all expenses are paid. The effects of making other assumptions as to the incidence of the expenses, &c., are easily seen.

NON-PARTICIPATING ASSURANCES.

Scope of the investigation.

The case of "non-profit" policies is considered first; since, apart from the intrinsic interest of the results, the effect of the various factors is more easily seen, and throws light upon the more complicated case of participating assurances.

The scope of the main investigation is as follows:

Ages at entry	...	30 and 50.
Classes of policy	...	Endowment Assurances, terms 15 and 25 years.
		Whole Life.
Valuation bases	...	Net premium valuations by:
		O^{M} 4%, $3\frac{1}{2}$% 3%, and $2\frac{1}{2}$%.
		$O^{M(5)}$ $3\frac{1}{2}$% and 3%.

Some special valuation bases set out on p. 171 are also examined.

Surplus from separate sources. It has not been thought necessary to publish the separate tables of the surplus from mortality and interest, from which the final tables were built up; since the characteristics of the surplus from these sources are well known, and indeed can be deduced from the general formulæ. Throughout, the tables printed have been cut down to the minimum necessary to illustrate the paper. For example, those relating to endowment assurances, term 15 years, have been omitted entirely, as they only showed the same characteristics as the 25 year term. The manuscript tables from which the abbreviated set, as printed, were prepared, are at the disposal of any member of the Institute to whom they are likely to be of service.

It may, however, be pointed out that since the O^{M} and $O^{M(5)}$ Tables are used in valuations as approximations to the $O^{[M]}$, much "profit" from mortality cannot be expected. If a table of Δq_n is examined, it will be seen that it is only in the case of policies entering after age 50 that a valuation by the O^{M} Table will show any appreciable "profit" from this source after the first quinquennium. For young ages at entry the surplus is very small, or negative, by the end of the second quinquennium, and remains so until age attained 70 is reached. If the $O^{M(5)}$ Table is used, the mortality "profit" is naturally much larger; but here also it decreases very rapidly after the first quinquennium and it seems probable that similar results will always appear when an office experiences select mortality and values by an aggregate table based upon similar data. As a matter of history it is also interesting to examine q_x by the H^{M} and O^{M} Tables and to note the great change in the amount and incidence of the mortality surplus caused by an alteration from the one valuation base to the other.

As regards interest, it may be noted that a decrease in the valuation rate has far more effect upon the surplus than a similar increase in the experience rate, since in the first case both $(V_{n-1}+\pi)$ and Δi are increased. It may also be pointed out that if both rates are increased by the same amount, the interest surplus is reduced; but that to some extent this is counterbalanced by the increase in the surplus from loading. [Although the surplus is reduced, the bonuses resulting therefrom may be increased, since the conversion factor $A_{\overline{n}|}^{-1}$ also increases with the valuation rate of interest.]

Results. The variations in the total surplus from all three sources, under different valuation conditions, are given in Tables I and II which differ only in the value of P used. Although only two ages at entry are given, the general characteristics remain the same for other ages at entry; the general effect of this being shown in Table IV. The characteristic features of Table I are brought out by Graph I showing the figures for age at entry 30 under an endowment assurance, term 25 years.

As shown in the Appendix, under a "net premium" valuation, the discounted, or the accumulated, value of the total surplus under a "non-participating" policy is quite independent of the valuation basis. The last only determines how much of this total amount is to be released year by year; and the rationale of the changes, which take place as the valuation basis is altered, will be readily seen, if it is remembered that approximately:

$$\text{Surplus O}^{\text{M}}\ 3\tfrac{1}{2}\ \% = \text{Surplus O}^{\text{M}}\ 3\ \%$$
$$+ [\underset{3\%}{\pi} - \underset{3\frac{1}{2}\%}{\pi}] \,.\, [1{\cdot}04] - {\cdot}005[V_{n-1} + \underset{3\frac{1}{2}\%}{\pi}]$$

That is, at the higher rate of interest the surplus is increased by a constant but relatively small amount of loading; and is diminished by loss of interest, which though small at first, rapidly increases in amount.*

These tables bring out the following points:

1. As the valuation basis becomes weaker, so is the surplus during the first ten years or so increased, and the surplus after that point diminished.
2. If the valuation basis is strengthened (*or weakened*) the office is in effect using (*or anticipating*) part of the surplus received (*to be received*) in the past (*future*) and the amount so used will be repaid in the future by an increase (*or a decrease*) in the surplus.
3. The very large amount of "profit" thrown off by the valuation bases usually adopted in this country during the later years of a policy's existence, as compared with the meagre or negative amounts during the early years.

* See Mr. George King's remarks, *J.I.A.*, vol. xxxii, p. 111; a formula there given suggesting a reason for the "point of equal surplus"; or in other words, the duration after which the various curves will intersect, being practically the same for all valuation bases. *Cf.* Graph I.

4. It is possible, in theory, to cause the surplus to be thrown off in any manner desired by varying the valuation basis. In practice, success would depend upon the degree of accuracy with which the future experience was gauged; although errors in the forecast would have less effect upon the general characteristics of the results than might be expected.
5. In theory, all valuations could be made to produce identical results as regards surplus by carrying forward suitable amounts. In practice, this would simply amount to a change in the valuation basis, except perhaps in name.

Some exceptional valuation bases. In Table III is shown the surplus arising from the following valuation bases:

$O^{[M]}$ Net Premium; $O^{M(10)}$ Ultimate; O^{M} using $O^{[M]}$ π; O^{M} using π_{x+1} (Sprague's method); $O^{M(5)}$ using $O^{[M]}$ π;

the valuation rate of interest being throughout 3 per-cent. The normal O^{M} and $O^{M(5)}$ figures are also inserted for comparison. Since these valuation bases are not in common use, figures are only given for one age at entry—30—and one class of policy (endowment assurance 25 years); but these are sufficient to show their main features, from which the general effect of using other combinations of the British Offices Tables may be deduced.

If a mixed valuation basis is used—as for example if an $O^{[M]}$ net premium is valued by O^{M} factors—an interesting, though academic, point arises as to the mortality surplus, since although the reserves approximate closely to those by an $O^{[M]}$ valuation the mortality surplus is practically the same as under an O^{M} valuation. This follows since the equation

$$\{V''_{n-1}+\pi_{[x]}\}\{1+i\}-q_{n-1}\,.\,\{1-V''_{n}\}=V''_{n}$$

holds good; the reserves being those on the mixed basis, and q_{n-1} being calculated by the O^{M} Table.

One possible hypothesis is that in valuing on a mixed basis we are in effect introducing a new aggregate table which gives net premium reserves equal to those on the mixed basis. If such a table were possible, large amounts of mortality surplus would probably arise owing to its eccentric nature.

A somewhat similar point arises whenever a net premium valuation is departed from, so that $V_0 \neq 0$, and a better explanation seems as follows. If π'' is the valuation premium

under the mixed basis, and π that for a "net premium" valuation, a fund of $[\pi-\pi''] \cdot \mathbf{a}_{\overline{t|}}$ (t=term of policy) comes into existence from, or for the benefit of, some source apart from the policy, at the moment when the latter is issued. That is, the office at the outset makes a loss (*or profit*) of $[\pi-\pi''] \cdot \mathbf{a}_{\overline{t|}}$ which is paid back (*for*) by annual amounts, which for the nth year consist of:

1. Loading surplus increased (*or diminished*) by

$$[\pi-\pi''][1+i']$$

2. Interest surplus increased (*or diminished*) by

$$[\pi-\pi''] \cdot a_{\overline{t-n|}} \cdot \Delta i$$

3. Mortality surplus diminished (*or increased*) by

$$[\pi-\pi''] \cdot \mathbf{a}_{\overline{t-n|}} \cdot \Delta q_{n-1}$$

over and above the surplus which would have been disclosed on the normal net premium basis. In Table III this initial loss has been accumulated until the end of the first year for the two bases where it occurs, in order to facilitate comparisons.

The chief points of interest in the results of Table III appear to be as follows:

1. The strain imposed by new business if a valuation is made by the $O^{[M]}$. This is of course due to the absence of any surplus from mortality and would be reduced by a more favourable experience rate.
2. A valuation based on $O^{[M]}$ net premiums and O^{M} factors appears to be better adapted to general use than the $O^{[M]}$ itself. The reserves are very similar but the strain of new business is reduced.
3. The $O^{M(10)}$ (ultimate) does not appear to differ greatly from, or to possess any advantages over, the $O^{M(5)}$ for valuation purposes.
4. Sprague's method. The method of treating expenses, and the use of a valuation rate of interest of 3 per-cent, are not fair to this method since they exaggerate its effects; but the figures serve to show how, as compared with other methods, the surplus for the first year is gained at the expense of the surplus for the next quinquennium, and in a lesser degree for the remainder of the policy's

existence. It appears probable that similar results would arise from any valuation making allowance for initial expenses; and that these methods will be most successful where a large "profit" from mortality can be counted upon throughout a policy's existence, or where conditions cause them to become "true" valuations.

Effect of massing contracts. So far only the surplus from one individual policy has been dealt with; and before drawing any general conclusions it is necessary to consider the effect of aggregating a large number of contracts of varying natures. When this is done the total surplus year by year will depend upon what—for want of a better term—may be called the "centre of gravity" of the total business; and if no new business were transacted, the total surplus would vary in a manner somewhat similar to the surplus from an individual policy of the mean age at entry, duration, and term unexpired. In practice, the accession of new business will retard the movements of the centre of gravity, and so of the surplus; so that if an office did an absolutely uniform new business, under exactly similar conditions from year to year, as in a "model office", the centre of gravity would eventually remain stationary, and the surplus would be identical from year to year.

It may be argued therefore that, under stable conditions, it is quite immaterial how the surplus is thrown off from each individual "non profit" policy; so long as the total amount at successive distributions is "correct", or in other words, meets the requirements of the particular office. In practice, however, it appears difficult to rely upon such steadiness, unless the surplus is "correct" in the case of each individual policy; since if this is not the case, any unusual changes in the flow of new business, &c., will cause changes in the centre of gravity which will be reflected in the surplus. Fluctuations in surplus cannot be eliminated and hence it does not appear desirable to add to the causes capable of producing them.

Conclusions. My general conclusions as regards "non profit" business are as follows:

1. A wide interest margin, such as exists under the stringent valuations usually adopted in this country, means that a very large proportion of the "profit" from a "non-participating" policy is not received until the later years of its existence, and that, even if the $3\frac{1}{2}$ per-cent

values of P are taken, a new policy may impose a strain upon an office for a period as long as ten years. This strain will be reduced as the experience rate of mortality falls below the $O^{[M]}$, and will be greatly increased if the new policy is debited with its initial expenses.

2. The value of a stringent valuation lies in the undivided surplus which is accumulated, since this forms a reserve to meet unforseen contingencies. Such reserves must be built up ; but in view of the difficulty of using them when a contingency arises, if they can only be released by weakening the valuation basis, it seems that such special reserves would be more valuable, if they were accumulated apart from the normal reserves.
3. Such special reserves apart, a wide interest margin is not essential for "non profit" assurances ; and the main factor in fixing the valuation basis should be that it will disclose the surplus in the manner most useful to that particular office. The nearer the basis approaches that of a "true" valuation, the nearer will the surplus, after perhaps the first year, approximate to a constant annual amount. Incidentally, the better too will the real position of the office be shown.
4. It appears desirable that, as a general rule, the surplus from a "non profit" policy should tend to be constant throughout its existence. If the greater part is to be divided amongst the participating policyholders, it would in theory be preferable if the surplus were given off from the non profit business in such a manner that its greatest assistance was received at the time when the surplus from the "with profit" business was least able to provide the usual bonus ; but in practice it seems impossible to arrange this. The theory is, however, an argument against using the same valuation basis for both "with" and "without" profit policies.
5. The "true" reserves should be the starting point of investigations into such questions as surrender values, changes in policies, &c. For example, if surrender values are based upon the reserves at 4 per-cent or $4\frac{1}{2}$ per-cent the office is automatically compensated for the loss of future surplus which would be caused by the withdrawal.

6. The arguments for and against the "net premium"—or indeed any method of valuation—depend upon the differences between the actual and valuation experience and not upon any inherent advantages or disadvantages of the method itself. The sole advantage of the "net premium" method, as such, is that $V_0 = 0$. In consequence, the reserves are consistent throughout the whole duration of the policy and negative values cannot appear.

PARTICIPATING POLICIES.

Since policyholders expect their share of the divisible surplus to be allotted in the shape of reversionary additions which increase, or at any rate remain constant, with successive distributions, it is necessary that the cash surplus under a participating policy should increase with its duration; the rate of increase required becoming very rapid as the policy approaches maturity.

From the preceding section, it will be obvious that such an increasing cash surplus can only be provided in practice by a wide interest margin; the magnitude of which can be seen from the following theoretical considerations.

Compound Bonus.—Dealing first with a compound bonus of h per annum, and assuming the same experience rates as in the preceding section, the net premium required to provide the sum assured and bonuses is :

$$\pi' = A_0^{j\%} \div \mathbf{a}_0^{4\%}$$

where $\mathbf{a}_0$ is calculated by the $O^{[M]}$ 4 per-cent and A_0 by the same table at rate j, where $1+j=(1\cdot04)\div(1+h)$.*

After n years, the reserve on a "true" basis will be :—

$$[1+h]^n . A_n^{j\%} - \pi' . \mathbf{a}_n^{4\%} \quad . \quad . \quad . \quad (3)$$

or

$$[1+h]^n . A_n^{j\%} - A_0^{j\%} \times \mathbf{a}_n^{4\%} \div \mathbf{a}_0^{4\%} . \quad . \quad . \quad (4)$$

Looked at from a slightly different standpoint the position is :

* When $j=2\frac{1}{2}\%$ $h=$ £1. 9*s*. 2*d*. % per annum.
3% $=$ £0. 19*s*. 5*d*. % per annum.
$3\frac{1}{2}\%$ $=$ £0. 9*s*. 8*d*. % per annum.

Reserve for sum assured and accrued bonus

$$[1+h]^{n-1} \,.\, A_n^{j\%} - \pi' \,.\, \mathbf{a}_n^{4\%}$$

\+ Surplus required to provide the new bonus

$$[1+h]^{n-1} \,.\, h \,.\, A_n^{j\%}$$

If the normal net premium valuation is made at rate j, using the $O^{[M]}$ Table, the reserve will be:

$$[1+h]^n \,.\, A_n - A_0 \times \mathbf{a}_n \div \mathbf{a}_0 \text{ all at rate } j \quad . \quad . \quad (5)$$

and since the function $\mathbf{a}_n \div \mathbf{a}_0$ decreases with the rate of interest, the reserves by formula 5 will always be greater than those by formula 3. That is, the surplus will be rather too small to provide the bonus at first, and larger than is required thereafter.

In practice a further modification—which may be called formula 6—will be introduced by the use of an aggregate table in the place of the $O^{[M]}$, the use of the O^M Table giving reserves which seem always to lie between those by formula 3 and formula 5.

Bearing in mind the modifications in the theoretical conditions which will occur in practice, it will be seen that an interest margin of Δi will provide a compound bonus of practically the same amount provided that:

1. The valuation table of mortality is a fairly close approximation to the experience rate.
2. The values of P are sufficient to provide the sum assured and bonuses on an experience basis.

As the reserves become weaker (*or stronger*) than those of a "bonus reserve" valuation upon the "true" basis, so will the cash surplus be greater (*or smaller*) than that required to provide a bonus of Δi during the first few years, and smaller (*greater*) than that required for this bonus thereafter.

These points are illustrated by the following table:

Endowment Assurance.

Age at Entry 30. *Death or* 25 *Years. Compound Bonus of* £1. 9*s*. 2*d*. *per-cent per annum.*

Duration	Reserves, including value of Accrued Bonuses			Total Bonuses	New Bonuses Added	"True" Basis Formula 3 Cash Surplus	Formula 6	
	Formula 3	Formula 5	Formula 6				Cash Surplus required	Actual Cash Surplus
1	3·69	3·97	3·71	1·46	1·46	·86	·86	·84
2	7·34	7·89	7·54	2·94	1·48	·89	·89	·74
3	11·10	11·91	11·48	4·44	1·50	·92	·92	·75
4	14·97	16·01	15·54	5·96	1·52	·96	·95	·78
5	18·97	20·24	19·75	7·51	1·55	1·00	·99	·82
10	41·11	43·26	42·85	15·58	1·66	1·18	1·18	1·09
15	68·00	70·43	70·21	24·27	1·79	1·42	1·42	1·49
20	101·32	103·20	103·11	33·60	1·92	1·70	1·70	1·95
25	143·63	143·63	143·63	43·63	2·06	2·06	2·06	2·51

Formula 3 ... $O^{[M]}$ $2\frac{1}{2}$ per-cent and 4 per-cent bonus reserve.
„ 5 ... $O^{[M]}$ $2\frac{1}{2}$ per-cent net premium reserves.
„ 6 ... O^{M} $2\frac{1}{2}$ per-cent net premium reserves.

Simple Bonus.—In the case of a simple bonus the rate of interest j in Formula 3 becomes:

$$1+j=[1\cdot04]\div[1+n\,.\,h]^{1/n}$$

and so increases with the duration of the policy. For example, if $h=$ £1. 10*s*. per-cent per annum, j will be:

$n=1$. £2. 9*s*. 3*d*. per-cent. $n=20$. £2. 12*s*. 11*d*. per-cent.
$n=10$. £2. 11*s*. 2*d*. per-cent. $n=50$. £2. 16*s*. 10*d*. per-cent.

That is, using formula 5, while a compound bonus of 30*s*. per-cent per annum would require a valuation rate of £2. 9*s*. per-cent and a level interest margin of 31*s*. per-cent; a simple bonus of similar amount would require a valuation rate rising from £2. 9*s*. per-cent to £2. 17*s*. per-cent and an interest margin decreasing from 31*s*. to 23*s*. Since a level valuation rate of interest gives far larger reserves than an increasing rate, it follows that a level interest margin of Δi will provide simple bonuses much smaller than Δi during the early years of a policy's existence and much greater than Δi thereafter.

These points are brought out in Table V showing the extent to which the bonuses actually earned on different valuation bases fall short of, or exceed, the assumed rate of 30*s*. per-cent per

annum. That is, it gives the values of $\frac{100 \cdot B_n^2}{A_n} - 1{\cdot}5$, the value of B_n^2 being that given by formula 2. The experience rates assumed, and scope of the investigation, are the same as for "non profit" policies; except that the values of P used are the net premiums, on the experience basis, for the sum assured plus a simple bonus of 30*s*. per-cent per annum.

Since Formula 2 includes the values of S_{n-1}, the accrued bonuses upon the basis of annual valuations, an approximation had to be made to obtain the values at quinquennial points, and it was assumed that S_{n-1} was equal to $[n-1] \times 1{\cdot}5$. A test showed that the effect of this approximation was very slight.

If the bonuses at each distribution were surrendered for their cash value calculated on the valuation basis, instead of being left in reversion as is assumed in formula 2, the "bonus surplus" would disappear; and, as pointed out by Mr. C. R. V. Coutts (*J.I.A.*, vol. xlii, p. 163), it would be necessary to lower the valuation rate of interest in order to maintain a given rate of bonus. If, however, the cash values were based upon the experience rates, or even lower values given, as would usually be the case, the profit made upon the surrender would increase the surplus at succeeding distributions and tend to obviate the necessity for reducing the valuation rate.

As might have been foreseen from theoretical considerations, none of the bases in Table V provide a simple bonus in an entirely satisfactory manner, and any other net premium valuation based on a level valuation rate and interest margin seems likely to be equally unsatisfactory. A valuation at $3\frac{1}{4}$ per-cent would approach closest to the figures required; and the table thus tends to confirm the old rule that a 30*s*. simple bonus requires a $\frac{3}{4}$ per-cent interest margin, if it is borne in mind that in practice the theoretical figures of Table V would be supplemented by quinquennial, in lieu of annual, distributions, surplus from miscellaneous sources and "non profit" business, and surplus from mortality and loading in excess of that assumed here. A level interest rate therefore, may give satisfactory results in practice as regards the business as a whole; though it will give widely varying results as regards individual policies.

Effect on different classes of policy. The following figures extracted from Table V show the great influence of the interest margin upon the relative bonuses under whole-life and endowment assurances:

Effect of Class of Policy upon rate of Reversionary Bonus earned.

Age at Entry 30.

Bonus under Whole-Life Policy less Bonus under Endowment Assurance 25 *Years.*

Duration	Valuation Basis		
	O^M 4 %	O^M 3 %	O^M 2½ %
1	·41	·23	·17
5	·18	·00	−·07
10	·11	−·14	−·24
15	·16	−·18	−·34
20	·21	−·23	−·43
25	·26	−·26	−·52

These results tend to confirm the conclusions arrived at by Messrs. H. J. Rietschel and H. H. Austin in their respective papers on the subject; and although, as pointed out by them, the results in any particular case depend mainly upon the office premiums and the allocation of the expenses, it seems probable that, at present, endowment assurances usually earn higher bonuses than whole-life policies. From the special point of view of this paper, the main interest of the figures lies in the fact that the differences arise solely from the valuation bases, and that if a "true" valuation were made both classes would earn the same bonus, namely, 30*s.* per-cent per annum.

The special reserves for unforeseen contingencies, mentioned under "non profit" policies, are of course, automatically provided in the case of participating assurances by the power to reduce, or to pass, a bonus.

Lack of time has hitherto prevented the investigation being extended to other methods of distribution; but the two dealt with are those most used in this country, and a comparison with them will show the main features which would be revealed by the investigation of other plans.

Conclusions.

My general conclusions in the case of "with profit" policies are:

1. A wide interest margin is a necessity in the case of "with profit" policies, if an ordinary net premium valuation is made, and a given rate of simple or compound bonus is to be maintained.
2. This method of making a net premium valuation, at a low rate of interest, is successful in practice because of

the approximation thereby obtained to a "true" or bonus reserve valuation based on the experience rates of mortality and interest.

3. This approximation tacitly assumes a compound bonus and in consequence is most successful with this method of distribution, or with methods of distribution the results of which approximate thereto. It is less well suited to a simple bonus; but in practice will work well under stable conditions, if the valuation rate of interest is higher than would be required for a compound bonus. It is easy to imagine methods of distribution for which it would be quite unsuitable.
4. In the case of a compound bonus this approximation gives results which are practically correct for each individual policy. In the case of a simple bonus, the results—though correct in the mass—may differ widely between one policy, and one class of policy, and another; and individually correct results can only be obtained by a "true" valuation. This, however, will only become important in practice if there is a violent change in the composition of the business, or the sources of surplus.

The results of the investigation may be summarized briefly as follows:

Summary.

The effect of the valuation basis adopted is not so much upon the total surplus produced by any particular policy as upon the time when, and the manner in which, this surplus is thrown off. Under a "non profit" policy, with a "true" valuation, the surplus would be practically constant year by year; and as the basis departs from this standard and becomes more stringent, the later in the life of the policy does the divisible surplus arise and the larger then becomes its amount. This process is necessarily accompanied by a reduction in the surplus during the earlier years; this reduction likewise becoming larger and more prolonged as the basis becomes more stringent.

Under a "participating" policy, a "true" valuation would produce the surplus necessary to set up the bonuses provided by the premiums; and as the basis becomes more stringent, so are the bonuses reduced during the early years of a policy's existence, and increased thereafter. It is also shown that the tendency of the methods usually adopted in this country is to give approximations to the "true" reserves under participating assurances.

APPENDIX AND TABLES.

Symbols and Terms Used.

Symbols when accented denote quantities calculated on the basis of the experience rates, non-accented symbols denoting those calculated on the valuation basis. Bonus is used throughout for reversionary bonus.

"True" valuation—a valuation based on the experience rates of mortality and interest, with a reserve for future bonuses in the case of participating policies.

All age suffixes, &c., have been omitted, the age at entry and class of policy being assumed constant throughout any given investigation, *e.g.*, V_n = Reserve at end of nth year; $p_n = l_{x+n+1} \div l_{x+n}$; $a_{\overline{t|}} = a_{x\overline{t|}}$, &c.

$\mathbb{P}$ = Office premium paid by the assured.

e = Actual expenses allocated to a policy.

$P = \mathbb{P} - e$.

π = Valuation net premium.

$\left.\begin{array}{l}\Delta i = i' - i \\ \Delta q = q - q'\end{array}\right\}$ Difference between the experience and valuation rates of interest and mortality respectively, a positive sign for either denoting a profit from that source.

B_n = Cash surplus arising during nth year from mortality and interest under a "non-profit" policy.

b_n = Corresponding to B_n but under a paid-up policy.

B^1_n = As B_n but from mortality, interest and loading.

B^2_n = As B^1_n but from a "with profit" policy, all previous amounts having been converted into reversionary bonuses by the valuation basis and added to the sum assured.

$K_n = B^2_n \div A_n$. Bonus added at end of nth year.

$S_n = K_1 + K_2 \ldots . K_n$. Total bonus additions up to and including nth year.

Formulæ Used.

"*Non-profit*" *Policies.*—Although the formulæ for the surplus arising from a policy are well known, it may be a convenience

if they are collected and restated here. These formulæ are independent of the class of policy so long as $p+q=1$.

At the end of the first policy year the total fund in respect of a policy for 1 is :—

$$l'_0 \,.\, [V_0+\pi]\,[1+i'] - d'_0 = l'_1 \,.\, V_1 + l'_1 \,.\, B_1$$

whence we have

$$[V_0+\pi]\,[1+i'] - q'_0 \,.\, [1-V_1] = V_1 + p'_0 \,.\, B_1$$

$$[V_0+\pi]\,[i'-i] + [1-V_1]\,[q_0-q'_0] = p'_0 \,.\, B_1$$

and generally

$$\{[V_{n-1}+\pi] \,.\, \Delta i + [1-V_n] \,.\, \Delta q_{n-1}\} \,.\, p'^{-1}_{n-1} = B_n$$

Bringing into account the premium paid by the assured, the office has at its disposal at the commencement of each year $\mathbb{P}-e=P$ and so has to find—or has at its disposal—$[P-\pi]$. Adding this amount, with the interest thereon, to B_n we have:

$$B^1_n = \{[V_{n-1}+\pi] \,.\, \Delta i + [1-V_n] \,.\, \Delta q_{n-1} + [P-\pi]\,[1+i']\} \,.\, p'^{-1}_{n-1} \quad . \quad . \quad (1$$

"Interest surplus" "Mortality surplus" "Loading surplus"

It will be seen that B^1_n is the cash surplus for each policy for 1 in force at the end of the nth policy year, or in other words for each policy surviving the year, on the assumption that the surplus of each year is carried to a separate fund and so does not affect the surplus in following years; no share of the surplus being taken by the policies becoming claims during the year. These conditions obtain in the case of "non-profit" policies.

If the portion of the surplus contributed by them, is allotted to and paid away with the claims arising during the year, formula 1 becomes:

$$[V_{n-1}+\pi] \,.\, \Delta i + [1-V_n] \,.\, \Delta q_{n-1} + [P-\pi]\,[1-i']$$

the factor $1/p'_{n-1}$ disappearing.

If no annual premium is payable we have:

$$b_n = \{A_{n-1} \,.\, \Delta i + [1 - A_n] \,.\, \Delta q_{n-1}\} \,.\, p'^{-1}_{n-1}$$

or giving the claims their share of the year's profits

$$b^2_n = \{A_{n-1} \,.\, \Delta i + [1 - A_n] \,.\, \Delta q_{n-1}\}$$

a result required in connection with "participating" policies.

It will be seen that Formula 1 is an example of Mr. G. J. Lidstone's "Variation Fund", vol. xxxix, p. 214.

As shown by the same author in vol. xxxii, p. 105, if the successive values of what I here term B_n are discounted back to the outset of the policy upon a "true" basis, the value at that point of the total profit to the office from a "non-profit" policy is:

$$D'^{-1}_0 \,.\, \Sigma B_n \,.\, D'_n = V_0 - V'_0 + [\pi - \pi'] \,.\, \mathbf{a}'_0$$

or adding the value of the "loading profit"

$$V_0 - V'_0 + [P - \pi'] \,.\, \mathbf{a}'_0$$

That is, if a net premium valuation is made, since $V_0 = V'_0 = 0$, the value to an office of a policy just placed on its books is independent of the valuation basis. The last only determines how much of this fixed total profit is to be released year by year, a result evident from general considerations. This would appear only to apply to "non-profit" policies, since the turning of a cash surplus into a reversionary bonus may set up a new source of profit.

If $V_0 \neq 0$ the above formula becomes

$$[P - \pi'] \,.\, \mathbf{a}'_0 - [\pi'' - \pi] \,.\, \mathbf{a}_0$$

where π'' is the valuation net premium. That is the total profit is diminished by the fund of $[\pi'' - \pi] \,.\, \mathbf{a}_0$ brought into existence (see slip 3) and to this extent is affected by the valuation basis.

At the end of the nth year after entry, the value of all subsequent surplus from a policy is:

$$V_n - V'_n + [P - \pi'] \,.\, \mathbf{a}'_n$$

"*With profit*" *policies.*—Hitherto only the question of cash surplus has been considered. When these cash amounts are turned into bonuses by the valuation basis, the question arises as to what share of the surplus is to be allotted to those policies becoming claims during the year. This has been fully discussed by Mr. G. J. Lidstone, vol. xxxii, p. 90, and following the lines there laid down it is assumed that claims receive their full share of the surplus arising in the year of death. That is p'^{-1}_{n-1} is eliminated from formula 1.

Assuming—as is done throughout the paper—annual valuations and distributions, at the end of the first year we have:

$$B^2_1 = p'_0 \,.\, B^1_1$$

This is converted into K_1 giving at the end of the second year:

$$[V_1 + \pi + K_1 \,.\, A_1][1 + i'] - q'_1 \,.\, [1 + K_1 + B^2_2]$$
$$= p'_1 \,.\, [V_2 + K_1 \,.\, A_2 + B^2_2]$$

whence

$$B^2_2 = p'_1 \,.\, B^1_2 + K_1 \,.\, b^2_2$$

and generally

$$B^2_n = [V_{n-1} + \pi] \,.\, \Delta i + [1 - V_n] \,.\, \Delta q_{n-1} + [P - \pi][1 + i']$$
$$+ S_{n-1} \,.\, \{A_{n-1} \,.\, \Delta i + [1 - A_n] \,.\, \Delta q_{n-1}\}$$

$$= p'_{n-1} \,.\, B^1_n + S_{n-1} \,.\, b^2_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of P in formula 2 will differ from that in formula 1, since the bonus loading has to be included.

Under similar conditions, K_n is, of course, the same, no matter whether it is declared as a simple or as a compound bonus. In the one case it is K_n per unit of the sum assured, and in the other $K_n \div [1 + S_{n-1}]$ per unit of the sum assured and previous bonuses.

TABLE I

Non-Participating Policy for £100. Total Surplus. Experience $O^{[M]}$ 4 *per-cent.*

$100\{[V_{n-1}+\pi].\Delta i+[1-V_n]\ \Delta q_{n-1}+[P-\pi][1\cdot04]\}.p'^{-1}_{n-1}$. *Values of* P *are the* $O^{[M]}$ 4 *per-cent Net Premiums.*

	Age at Entry 30						Age at Entry 50						
	Valuation Basis												
n	O^M 4%	O^M 3½%	O^M 3%	O^M 2½%	$O^{M(5)}$ 3½%	$O^{M(5)}$ 3%	O^M 4%	O^M 3½%	O^M 3%	O^M 2½%	$O^{M(5)}$ 3½%	$O^{M(5)}$ 3%	n
	Endowment Assurance. Death or 25 years.												
1	·26	·10	−·06	−·23	·19	·03	·55	·42	·27	·12	·43	·29	1
5	·01	−·10	−·20	−·31	−·06	−·17	·08	·00	−·08	−·17	·01	−·07	5
10	−·07	−·09	−·11	−·13	−·10	−·13	−·19	−·19	−·19	−·18	−·19	−·19	10
15	−·05	·02	·09	·17	−·02	·05	−·20	−·11	−·02	·09	−·12	−·03	15
20	−·03	·15	·33	·52	·10	·28	−·19	·01	·21	·42	−·02	·19	20
25	−·02	·30	·62	·93	·24	·56	−·21	·15	·50	·85	·12	·48	25
	Whole Life.												
1	·27	·18	·07	−·06	·27	·16	·56	·45	·33	·20	·46	·34	1
5	·02	−·06	−·14	−·23	−·02	−·10	·10	·03	−·04	−·12	·04	−·04	5
10	−·07	−·11	−·15	−·20	−·11	−·15	−·18	−·19	−·20	−·20	−·19	−·20	10
15	−·05	−·06	−·06	−·06	−·09	−·09	−·19	−·14	−·08	−·02	−·15	−·09	15
20	−·04	−·01	·03	·08	−·04	·00	−·18	−·06	·06	·19	−·08	·04	20
25	−·03	·04	·13	·22	·00	·09	−·17	·01	·20	·38	−·02	·17	25
35	−·01	·16	·33	·52	·10	·28	−·17	·14	·45	·77	·11	·42	35
45	·02	·28	·55	·82	·21	·48	...	...	...	...	...	...	...
55	·03	·41	·78	1·15	·33	·71	...	...	...	...	...	...	...

TABLE II.

Non-Participating Policy for £100. Total Surplus. Experience $O^{[M]}$ 4 *per-cent.*

$100\{[V_{n-1}+\pi].\Delta i+[1-V_n].\Delta q_{n-1}+[P-\pi][1{\cdot}04]\}.p'^{-1}_{n-1}$. *Values of* P *are the* $O^{[M]}$ $3\frac{1}{2}$ *per-cent Net Premiums.*

	Age at Entry 30						Age at Entry 50						
	Valuation Basis												
n	O^M 4%	O^M 3½%	O^M 3%	O^M 2½%	$O^{M(5)}$ 3½%	$O^{M(5)}$ 3%	O^M 4%	O^M 3½%	O^M 3%	O^M 2½%	$O^{M(5)}$ 3½%	$O^{M(5)}$ 3%	n
	Endowment Assurance. Death or 25 years.												
1	·43	·28	·11	−·06	·36	·20	·72	·58	·43	·28	·59	·45	1
5	·18	·07	−·03	−·14	·11	·00	·25	·17	·08	·00	·18	·09	5
10	·11	·08	·06	·05	·07	·05	−·02	−·02	−·02	−·02	−·03	−·02	10
15	·13	·19	·27	·35	·15	·23	−·03	·06	·15	·25	·05	·14	15
20	·14	·32	·50	·69	·27	·45	−·02	·18	·38	·59	·16	·36	20
25	·16	·48	·79	1·10	·41	·73	−·03	·33	·68	1·03	·30	·66	25
	Whole Life.												
1	·38	·28	·17	·05	·37	·27	·70	·58	·46	·33	·60	·48	1
5	·13	·05	−·03	−·12	·09	·01	·24	·17	·09	·02	·18	·10	5
10	·04	·00	−·04	−·09	·00	−·05	−·04	−·05	−·06	−·06	−·05	−·06	10
15	·06	·05	·05	·05	·02	·02	−·05	·00	·06	·13	−·01	·05	15
20	·07	·10	·14	·19	·07	·11	−·03	·08	·21	·33	·06	·19	20
25	·08	·15	·24	·33	·11	·20	−·02	·16	·34	·53	·13	·32	25
35	·10	·27	·44	·63	·22	·39	−·00	·31	·62	·93	·28	·59	35
45	·13	·40	·66	·94	·33	·60	...	...	...	...	...	...	...
55	·17	·54	·91	1·29	·46	·84	...	...	...	...	...	...	...

Table III.

Non-Participating Policy for £100.
Age at Entry 30. *Death or* 25 *years.*
Some Special Valuation Bases. *Total Surplus.*
Experience $O^{[M]}$ 4 *per-cent.* P *Net Premiums on basis of* $O^{[M]}$ 4 *per-cent.*
Valuation Rate of Interest 3 *per-cent.*

n	O^M	$O^{M(5)}$	$O^{[M]}$	O^M Select π	$O^{M(5)}$ Select π	$O^{M(10)}$ Ultimate	O^M Sprague's Method π_{x+1}	n
1	−·06	·03	−·32	−·30	−1·02	·03	2·61	1
2	−·20	−·12	−·29	−·18	− ·04	−·12	−·39	2
3	−·22	−·16	−·26	−·20	− ·08	−·16	−·42	3
4	−·21	−·17	−·23	−·20	− ·08	−·17	−·41	4
5	−·20	−·17	−·20	−·18	− ·08	−·17	−·40	5
10	−·11	−·13	−·05	−·09	− ·04	−·12	−·31	10
15	·09	·05	·14	·11	·14	·05	−·10	15
20	·33	·28	·36	·35	·36	·28	·15	20
25	·62	·56	·63	·63	·63	·55	·44	25

Table IV.

Non-Participating Policy for £100.
Death or 25 *years.*
Effect of Age at Entry on Total Surplus.
Experience and P $O^{[M]}$ 4 *per-cent.* *Valuation* O^M 3 *per-cent.*

n	Age at Entry				n
	20	30	40	50	
1	−·13	−·06	·08	·27	1
2	−·26	−·20	−·08	·08	2
3	−·28	−·22	−·12	·00	3
4	−·27	−·21	−·14	−·05	4
5	−·25	−·20	−·15	−·08	5
10	−·11	−·11	−·14	−·19	10
15	·13	·09	·05	−·02	15
20	·39	·33	·28	·21	20
25	·68	·62	·57	·50	25

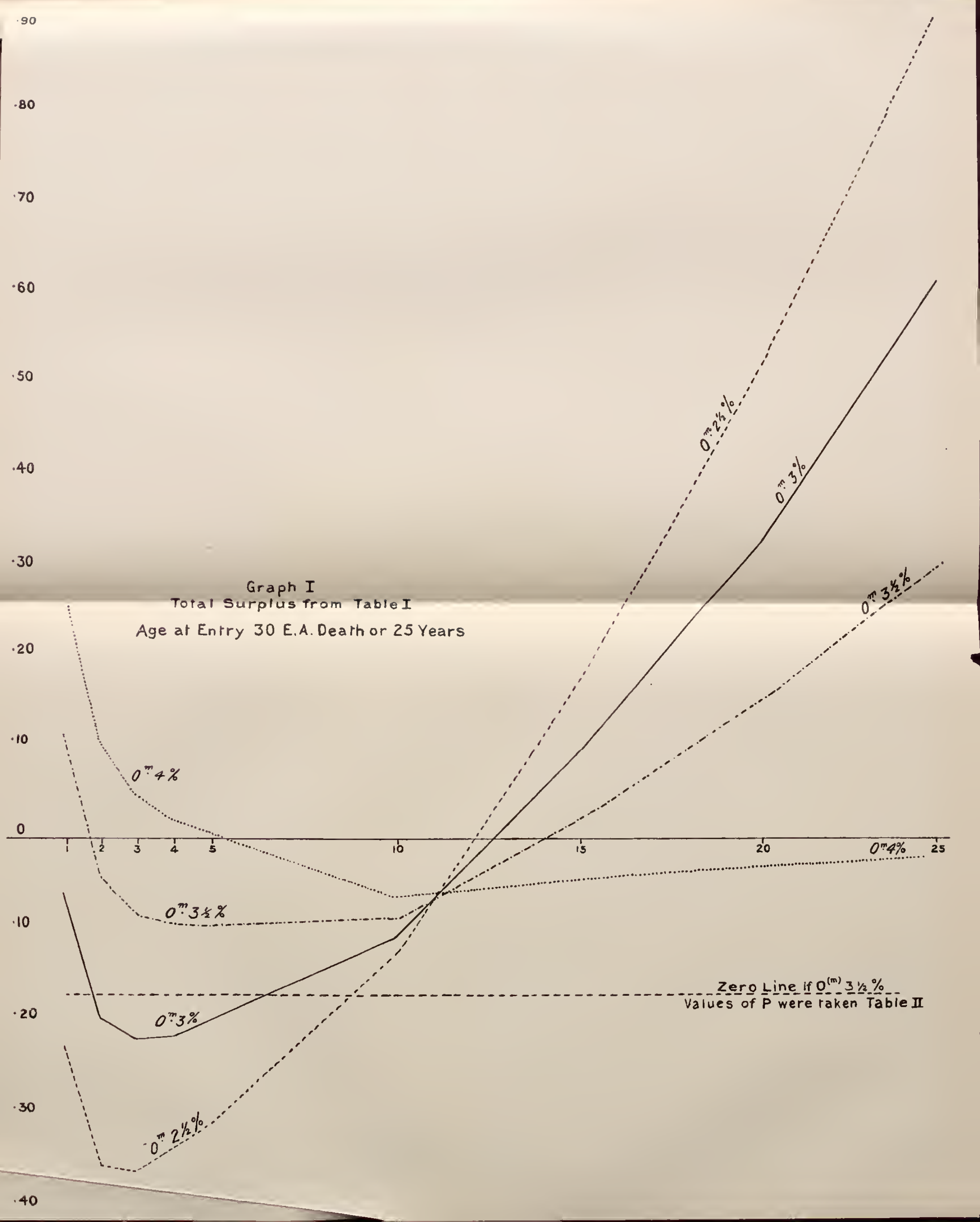
TABLE V.

Participating Policy for £100.

Difference between Bonus actually earned by Policy and Simple Bonus of 30s. provided by Premiums.

$\frac{100B^2_n \text{ (Form}^{l.} 2)}{A_n} - 1{\cdot}500.$ *Experience* $O^{[M]}$ 4 *per-cent.* P = *Net Premiums on basis of* $O^{[M]}$ 4 *per-cent.*

	Age at Entry 30						Age at Entry 50						
	Valuation Basis												
n	O^M 4%	O^M 3½%	O^M 3%	O^M 2½%	$O^{M(5)}$ 3½%	$O^{M(5)}$ 3%	O^M 4%	O^M 3½%	O^M 3%	O^M 2½%	$O^{M(5)}$ 3½%	$O^{M(5)}$ 3%	n
Endowment Assurance. Death or 25 years.													
1	1·22	·64	·13	−·32	·80	·29	1·46	1·00	·58	·19	1·03	·60	1
5	·38	·07	−·20	−·44	·14	−·14	·38	·16	−·05	−·22	·17	−·03	5
10	−·03	−·09	−·14	−·18	−·11	−·16	−·26	−·26	−·25	−·23	−·26	−·25	10
15	−·23	−·09	·05	·20	−·14	·00	−·42	−·24	−·07	·11	−·26	−·08	15
20	−·42	−·09	·23	·54	−·15	·17	−·54	−·20	·13	·46	−·23	·11	20
25	−·59	−·10	·39	·87	−·16	·33	−·70	−·20	·30	·80	−·23	·28	25
Whole Life.													
1	1·63	·96	·37	−·15	1·17	·57	1·53	1·08	·67	·28	1·10	·69	1
5	·56	·15	−·20	−·51	·26	−·10	·41	·19	−·01	−·20	·21	·00	5
10	·09	−·11	−·27	−·42	−·12	−·28	−·26	−·28	−·29	−·29	−·27	−·28	10
15	−·07	−·11	−·13	−·14	−·16	−·18	−·39	−·26	−·13	−·01	−·27	−·14	15
20	−·21	−·11	·00	·11	−·16	−·05	−·45	−·20	·04	·28	−·23	·02	20
25	−·33	−·10	·13	·35	−·16	·07	−·51	−·15	·20	·54	−·18	·17	25
35	−·51	−·07	·37	·78	−·14	·29	−·59	−·06	·45	·96	−·09	·43	35
45	−·61	−·01	·58	1·16	−·09	·51	...	...	...	...	...	...	...
55	−·68	·05	·77	1·48	−·02	·71	...	...	...	...	...	...	...


Graph I
Total Surplus from Table I
Age at Entry 30 E.A. Death or 25 Years
·90
·80
·70
·60
·50
·40
·30
·20
·10
0
·10
·20
·30
·40
1
2
3
4
5
10
15
20
25
O^m 2½%
O^m 3%
O^m 3½%
O^m 4%
Zero Line if O^(m) 3½%
Values of P were taken Table II

Abstract of the Discussion.

Mr. S. J. ROWLAND said the subject of investigation was to what extent valuation, and valuation alone, would produce positive or negative surpluses, apart from the actual experience of the office. Assuming a closed fund, from which no distribution of surplus was made, the amount of the fund from time to time would depend entirely upon the premiums and interest received and the claims paid, and, when the fund had finally worked itself out, the positive or negative balance at the end would depend only on the actual experience of the office ; the valuation basis would not affect in any way the actual profits made by the office. When the profits shown by valuation were distributed, rather different considerations came in. For example, if the valuation basis gave a large amount of negative reserves, and these were not eliminated, there would be a surplus merely as the result of the valuation basis, and, if that surplus were distributed, then, of course, the office would sustain a corresponding loss later on. He was not quite certain that the author's method of dealing with the strain in the first year for non-participating policies was on the lines usually adopted in practice, because in many cases the office premium, after allowing for expenses, was less than the net premium by the valuation basis, and it was therefore necessary to use either the Office premium or some percentage of it as the valuation premium, and of course that threw very considerable strain on the first year.

The justification for valuation at a low rate of interest in the case of with profit policies seemed to him to be that it gave an approximation to a bonus reserve method and was much simpler. If that were so, it must be wrong for non-participating business. He thought there should be a fairly wide margin between the valuation rate of interest for participating policies and the rate for non-participating policies, and, if the office wished to make an additional reserve in the case of non-participating policies, such additional reserve should be entirely separate, because, if it were included in the ordinary valuation reserve, it was almost impossible to use it when it was needed.

Mr. R. C. SIMMONDS thought that the author appeared to have passed too lightly over a very important element—the question of expenses. If he had dealt with the problems of practical valuation as they came before the actuary, he thought there would have been a recognition of three specially important points. The first was that the office received only a level periodical premium ; secondly, in practically every case the expenses were very heavy in the early years, in the first year at any rate, and thirdly, that a uniform reversionary bonus required an increasing cash surplus. It was clear from general considerations that, expenses being distributed as they were, the putting up of an ordinary net premium reserve imposed a very heavy strain, and that strain was increased still more when an attempt was made to allow for the increasing reversionary bonus by means of a lower rate of interest.

For practical purposes it seemed to him necessary to consider

not only an individual contract as such but the business in the aggregate ; not how the surplus was dealt with in respect of any individual policy so much as whether the profit released on account of the old policies compensated for the money which had to be locked up on account of new business. Therefore he thought it might have been useful had the investigation extended to groups of policies, and had an attempt been made to take what was, at the moment, quite a common case, that of an office which had done a relatively large amount of new business in the last year or two. If the results of such an investigation were available, it would be possible to know what, after all, must be known before any surplus could be distributed with confidence, namely, whether the valuation methods were or were not giving a reasonable practical result in the aggregate.

MR. A. E. BROWN was not in favour of making the necessary reserve for non-participating business on a true basis and of making what might be called a safety margin reserve separately. Where the usual practice was adopted of making a net premium valuation on a basis which left some margin for safety, one knew one had a special reserve adapted to meet contingencies. He did not see how it was possible to provide such a reserve by any arbitrary method.

At the present time, having regard to the fact that appreciation or depreciation of investments was in effect the same as an increased or decreased rate of interest, it seemed to him impossible to estimate accurately the future rate of interest, and any bonus reserve valuation would require frequent adjustment to bring the rates into line with actual experience, and would in consequence lose all its advantages. He thought, therefore, the conclusion to be drawn from the paper was that no matter what method of valuation was employed, any fixed method of distribution was certain, sooner or later, to cause embarrassment as business expanded or contracted, and that wherever possible measures should be taken to adopt a contribution method of distribution.

MR. A. S. HOLNESS thought that it was the common experience of most offices that their non-participating business followed much the same lines as the participating business, and as a result any surplus arising from the non-participating business, if it was the practice of the office to distribute it mainly to with-profit policyholders, would be simply a constant addition to the rate of bonus earned on the with-profit policies. There was something to be said for investigating the effect of valuation bases on individual policies. A basis of valuation such that the withdrawal of new business would entirely upset the incidence of the surplus would be an unsatisfactory basis, and it was desirable, therefore, to keep in view, the effect of the basis on individual policies. He thought it should not be overlooked in considering the author's paper that the profits or losses which he showed at any particular valuation were influenced by the valuation basis adopted at previous valuations, and it was rather interesting to compare the valuation reserves on the various bases with the reserves which would be

required had no valuation on those bases been made but had the fund been allowed to accumulate on the experience basis. The results came out very differently perhaps from what might be expected from the paper.

Mr. W. P. ELDERTON said there were a few more or less practical points that occurred to him in reading the paper. Taking the case of two offices, one of which gave a uniform bonus and the other a compound bonus, and assuming that those two offices were earning the same rate of interest and had the same expenses, so that they gave equally advantageous results, what happened in competition ? The compound bonus office took credit for its lower valuation rate and talked of extra reserves and so on. As a matter of fact, it ought to defend its low valuation rate not on the ground of stringency, but because it threw up approximately the right surplus from year to year ; in other words, its argument was based on a misunderstanding, and he was afraid actuaries were responsible for that misunderstanding. Another point arose out of the author's treatment of non-profit insurance. He wondered how many actuaries when they made a very stringent valuation of non-profit business appreciated that there were possibly real dangers ahead. They began by hiding reserves, cutting something out of surplus. Gradually, as the author had shown, that came back. Taking the case of an office, a common case in 1913, which had practically all with-profit business, and supposing that it did a large without-profit business, it took the reserve out of profit and showed too little surplus. It gave the with-profit policyholders less than they ought to have had. As time went on and the profit came back it increased the bonus on its with-profit policies. That attracted with-profit business and the without-profit business fell off. One actuary perhaps had used up a profit which he himself created, and left his successor to bear the criticism. Possibly a half-hearted appreciation of that was the reason why actuaries were frightened of doing more than a certain amount of new business. If they could not do with as much good new business as they could obtain, which was wrong—the new business or the valuation ?

The PRESIDENT said it was very pleasing once again, after the events of the last few years, to have before the Institute a paper which turned largely upon bonus prospects and the provision which ought to be made for them. It indicated, at any rate to those who were policyholders, that a brighter future was in store.

Mr. MALTBY, in reply, pointed out that it was possible to change his tables so as to give effect to varying expenses in about five minutes. It was simply a matter of a constant deduction for the first year and a constant addition thereafter. With regard to the question of making a valuation of non-profit business on a true basis with a reserve for future contingencies, he would suggest that the most serious contingency for which provision had to be made was war in one form or another, and that was better provided for by a reserve for depreciation of assets than by means of the valuation basis.

On certain Formulas of Approximate Summation and Integration. *By* J. F. STEFFENSEN, D.Phil., *Corresponding Member of the Institute of Actuaries.*

THE remainder form of Newton's formula of interpolation with divided differences has been known for at least 37 years*; yet actuaries have taken hardly any notice of the complete formula which ought to be one of their principal tools. It will, therefore, not be superfluous to begin by reproducing the very elementary proof from first principles.

As regards the class of functions considered in this paper, we assume that the function possesses, throughout the range of arguments in question, a *continuous* differential coefficient of the highest order of which use is made in deriving each particular formula under consideration. This restriction is not very serious; it is, for instance, not necessary, that the function should possess differential coefficients of every order; much less, that it should be a polynomial. On the other hand, care is necessary in applying interpolation formulas to functions obtained by joining together pieces of different analytical functions, not to speak of tables graduated by graphic or mechanical methods.

Let the function be $f(x)$; the divided differences of the 1st, 2nd, . . . rth order are, then, defined by the equations

$$\left.\begin{aligned} f(x_0, x_1) &= \frac{f(x_0)-f(x_1)}{x_0-x_1} \\ f(x_0, x_1, x_2) &= \frac{f(x_0, x_1)-f(x_1 . x_2)}{x_0-x_2} \\ \vdots \quad & \quad \vdots \quad \vdots \quad \vdots \\ f(x_0, \ldots x_r) &= \frac{f(x_0, \ldots x_{r-1})-f(x_1, \ldots x_r)}{x_0-x_r} \end{aligned}\right\} \quad . \quad . \quad (1)$$

provided that the arguments $x_0, x_1, \ldots x_r$ are all different. If two or more of them coincide, differential coefficients are evidently introduced. The value of $f(x_0, \ldots\ldots x_r)$ is independent of the order in which $x_0, \ldots x_r$ are taken.

If in (1) we replace $x_0, x_1, \ldots x_r$ by $x, a_0, a_1, \ldots a_n$, we

*Laurent: Traité d'Analyse (1885), vol. i, pp. 104, 107. The next real progress is due to Dr. Jensen, who has given new expressions for the divided differences and for the remainder-term. See J. L. W. V. Jensen: Sur une expression simple du reste dans la formule d'interpolation de Newton (Bulletin de l'Académie Royale de Danemark, 1894); and further historical notes in a recent paper by Glenny Smeal: On Direct and Inverse Interpolation by Divided Differences (Proceedings of the Edinburgh Mathematical Society, vol. xxxviii).

obtain a system that may be written

$$\left.\begin{aligned} (x-a_0)f(x, a_0) &= f(x)-f(a_0) \\ (x-a_1)f(x, a_0, a_1) &= f(x, a_0)-f(a_0, a_1) \\ \vdots \quad\quad \vdots \quad\quad & \vdots \quad\quad \vdots \\ (x-a_n)f(x, a_0, \ldots a_n) &= f(x, a_0, \ldots a_{n-1})-f(a_0, \ldots a_n) \end{aligned}\right\} . \quad (2)$$

From this we obtain in succession

$$\begin{aligned} f(x) &= f(a_0)+(x-a_0)f(x, a_0) \\ &= f(a_0)+(x-a_0)f(a_0, a_1)+(x-a_0)(x-a_1)f(x, a_0, a_1) \\ &= \ldots\ldots\ldots \end{aligned}$$

and generally

$$\begin{aligned} f(x) &= f(a_0)+(x-a_0)f(a_0, a_1)+(x-a_0)(x-a_1)f(a_0, a_1, a_2)+\ldots \\ &\quad +(x-a_0)(x-a_1)\ldots(x-a_{n-1})f(a_0, a_1, \ldots a_n)+\mathrm{R} \quad (3) \end{aligned}$$

where

$$\mathrm{R}=(x-a_0)(x-a_1)\ldots(x-a_n)f(x, a_0, a_1, \ldots a_n) \quad . \quad . \ (4)$$

This is Newton's formula with divided differences. Let us, for a moment, write it

$$f(x)=\phi(x)+\mathrm{R}(x) \quad . \quad . \quad . \quad . \quad . \quad . \ (5)$$

denoting by $\phi(x)$ the polynomial of degree n, and by $\mathrm{R}(x)$ the remainder.

As $\mathrm{R}(x)$ vanishes at the $n+1$ points $a_0, a_1, \ldots a_n$, its differential coefficient $\mathrm{R}'(x)$ vanishes at least n times, and therefore $\mathrm{R}''(x)$ at least $n-1$ times, &c., and finally $\mathrm{R}^{(n)}(x)$ at least once, in any interval containing $a_0, a_1, \ldots a_n$. Now, let ξ be such a value that $\mathrm{R}^{(n)}(\xi)=0$. Then, by (5)

$$\begin{aligned} 0=\mathrm{R}^{(n)}(\xi) &= f^{(n)}(\xi)-\phi^{(n)}(\xi) \\ &= f^{(n)}(\xi)-n!\,f(a_0, a_1, \ldots a_n) \end{aligned}$$

whence $$f(a_0, a_1, \ldots a_n)=\frac{1}{n!}f^{(n)}(\xi) \quad . \quad . \quad . \quad . \ (6)$$

This formula is perfectly general, the quantities a_v are any $n+1$ arguments, not necessarily all different, and ξ is an unknown quantity situated between the smallest and largest of the quantities a_v.

It follows at once from (6), that

$$f(x,\, a_0,\, a_1,\, \ldots\, a_n) = \frac{1}{(n+1)\,!} f^{(n+1)}(\xi)$$

where ξ now means an intermediate quantity between $x, a_0, a_1, \ldots a_n$, so that (4) may be written

$$\mathrm{R} = (x-a_0)(x-a_1)\,\ldots\,(x-a_n)\frac{f^{(n+1)}(\xi)}{(n+1)\,!} \quad . \quad . \quad . \quad (7)$$

From (3) and (4) it is easy to derive most of the interpolation-formulas usually employed, with their remainder-terms. We need not go into this here, but confine ourselves to the formulas of which use is made later on.

Putting first $a_v = v$, we obtain immediately by (7)

$$f(x) = \sum_{v=0}^{n} \frac{x^{(v)}}{v\,!}\Delta^v f(0) + \frac{x^{(n+1)}}{(n+1)\,!} f^{(n+1)}(\xi) \quad . \quad . \quad . \quad (8)$$

Next, putting $a_{2r} = r$, $a_{2r-1} = -r$, we find

$$\left.\begin{aligned} f(x) = \sum_{r=0}^{k} &\frac{(x+r)^{(2r)}}{(2r)\,!}\Delta^{2r} f(-r) \\ &+ \sum_{r=0}^{k-1} \frac{(x+r)^{(2r+1)}}{(2r+1)\,!}\Delta^{2r+1} f(-r-1) + \mathrm{R} \\ \mathrm{R} = (x+k)^{(2k+1)} &f(x,\, 0,\, \pm 1,\, \ldots\, \pm k) \end{aligned}\right\} \quad . \quad . \quad (9)$$

and, for $a_{2r} = -r$, $a_{2r-1} = r$,

$$\begin{aligned} f(x) = \sum_{r=0}^{k} &\frac{(x+r-1)^{(2r)}}{(2r)\,!}\Delta^{2r} f(-r) \\ &+ \sum_{r=0}^{k-1} \frac{(x+r)^{(2r+1)}}{(2r+1)\,!}\Delta^{2r+1} f(-r) + \mathrm{R} \quad . \quad . \quad (10) \end{aligned}$$

where R has the same value as in (9).

Now, let δ denote the central difference, that is

$$\delta f(x) = f\left(x+\frac{1}{2}\right) - f\left(x-\frac{1}{2}\right)$$

so that $\Delta^{2r} f(-r) = \delta^{2r} f(0)$. Further, let $x^{[n]}$ be the *central factorial*, defined by $x^{[0]} = 1$, $x^{[1]} = x$, and generally

$$\left.\begin{aligned} x^{[n]} &= x\left(x+\frac{n}{2}-1\right)\left(x+\frac{n}{2}-2\right)\ldots\left(x-\frac{n}{2}+1\right) \\ &= x\left(x+\frac{n}{2}-1\right)^{(n-1)} \end{aligned}\right\} \tag{11}$$

so that $\delta x^{[n]} = nx^{[n-1]}$. We shall evidently have

$$\left.\begin{aligned} x^{[2r]} &= x^2(x^2-1)(x^2-4)\ldots(x^2-\overline{r-1}^2) \\ x^{[2r+1]} &= x\left(x^2-\frac{1}{4}\right)\left(x^2-\frac{9}{4}\right)\ldots\left(x^2-\frac{(2r-1)^2}{4}\right) \end{aligned}\right\} \tag{12}$$

so that $x^{[2r]}$ is an even, $x^{[2r+1]}$ an odd function of x.

In this notation we have

$$\begin{aligned} \frac{\mathrm{R}(x)+\mathrm{R}(-x)}{2} &= (x+k)^{(2k+1)} \cdot \frac{f(x, 0, \pm 1, \ldots \pm k) - f(-x, 0, \pm 1, \ldots \pm k)}{2} \\ &= x^{[2k+2]} \cdot \frac{f(x, 0, \pm 1, \ldots \pm k) - f(-x, 0, \pm 1, \ldots \pm k)}{2x} \\ &= x^{[2k+2]} f(\pm x, 0, \pm 1, \ldots \pm k) \\ &= \frac{x^{(2k+2)}}{(2k+2)\,!} f^{(2k+2)}(\xi) \end{aligned}$$

so that we obtain from (9) and (10)

$$\begin{aligned} \frac{f(x)+f(-x)}{2} &= \sum_{r=0}^{k} \frac{(x+r)^{(2r)}}{(2r)\,!} \Delta^{2r} f(-r) \\ &\quad + \frac{1}{2}\sum_{r=0}^{k-1} \frac{(x+r)^{(2r+1)}}{(2r+1)\,!} \Delta^{2r+1}[f(-r-1) - f(-r)] + \mathrm{R} \\ &= \sum_{r=0}^{k} \frac{(x+r)^{(2r)}}{(2r)\,!} \Delta^{2r} f(-r) \\ &\quad - \frac{1}{2}\sum_{r=0}^{k-1} \frac{(x+r)^{(2r+1)}}{(2r+1)\,!} \Delta^{2r+2} f(-r-1) + \mathrm{R} \end{aligned}$$

which is easily reduced to

$$\frac{f(x)+f(-x)}{2} = \sum_{r=0}^{k} \frac{x^{[2r]}}{(2r)\,!} \delta^{2r} f(0) + \frac{x^{[2k+2]}}{(2k+2)\,!} f^{(2k+2)}(\xi) \quad . \quad . \quad (13)$$

We are now in a position to proceed to derive formulas of summation and integration. As regards the notation, ξ

denotes always a quantity intermediate between the arguments employed in the formula, and R the remainder term. It must, therefore, not be expected that two quantities, both denoted by ξ or by R, are identical.

If, in the identity

$$\int_0^m f(x)dx = \sum_{r=0}^{m-1}\int_0^1 f(t+r)dt \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we express $f(t+r)$ by a series of the form (8)

$$f(t+r) = \sum_{v=0}^{n}\frac{t^{(v)}}{v\,!}\Delta^v f(r) + \frac{t^{(n+1)}}{(n+1)\,!}f^{(n+1)}(\xi) \quad . \quad . \quad (15)$$

we obtain

$$\int_0^m f(x)dx = \sum_{r=0}^{m-1}\int_0^1 \sum_{v=0}^{n}\frac{t^{(v)}}{v\,!}\Delta^v f(r)dt + \mathrm{R} \quad . \quad . \quad (16)$$

$$\mathrm{R} = \sum_{r=0}^{m-1}\int_0^1 \frac{t^{(n+1)}}{(n+1)\,!}f^{(n+1)}(\xi)dt \quad . \quad . \quad . \quad (17)$$

where it must be remembered that ξ depends on t and r.

If we write

$$\mathrm{L}_v = \int_0^1 \frac{t^{(v)}}{v\,!}dt \ (\mathrm{L}_0 = 1) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(16) becomes

$$\left.\begin{aligned}\int_0^m f(x)dx &= \sum_{v=0}^{n}\mathrm{L}_v\sum_{r=0}^{m-1}\Delta^v f(r) + \mathrm{R}\\ &= \sum_{r=0}^{m-1}f(r) + \sum_{v=1}^{n}\mathrm{L}_v\Delta^{v-1}[f(m) - f(0)] + \mathrm{R}\end{aligned}\right\} \quad . \quad . \quad . \quad (19)$$

In order to simplify the remainder-term, we must apply the Theorem of Mean Value which may be expressed thus:

Let $\phi(x)$ and $\psi(x)$ be two functions which, in the interval of integration, satisfy the conditions

$$\phi(x) > 0; \quad g < \psi(x) < \mathrm{G};$$

then

$$g\int_a^b \phi(x)dx < \int_a^b \psi(x)\phi(x)dx < \mathrm{G}\int_a^b \phi(x)dx$$

or, if $\psi(x)$ is continuous,

$$\int_a^b \psi(x)\phi(x)dx = \psi(\xi)\int_a^b \phi(x)dx \ (a < \xi < b) \quad . \quad . \quad (20)$$

The proof is obvious, but can be found in any text-book

on analysis. It was assumed that $\phi(x) > 70$; but (20) evidently holds provided $\phi(x)$ *does not change its sign* in the interval $a < x < b$.

Applying this to (17), we notice that $t^{(n+1)}$ does not change its sign for $0 < t < 1$, so that

$$R = \sum_{r=0}^{m-1} f^{(n+1)}(\xi) L_{n+1} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

ξ having no more the same value as in (17), but being situated between the same limits as before.

Now, $\frac{1}{m} \sum_{r=0}^{m-1} f^{(n+1)}(\xi)$ is a mean value of $f^{(n+1)}(x)$ which may be called $f^{(n+1)}(\xi)$, all we know of the new ξ being that it is situated between 0 and $m+n-1$. We therefore have

$$R = mL_{n+1} f^{(n+1)}(\xi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

or finally, combining this formula with (19),

$$\int_0^m f(x)dx = \sum_{v=0}^{m-1} f(v) + \sum_{v=1}^{n} L_v[\Delta^{v-1} f(m) - \Delta^{v-1} f(0)] \\ + mL_{n+1} f^{(n+1)}(\xi) \quad . \quad . \quad (23)$$

The coefficients L_v are easily calculated. We have for sufficiently small values of t

$$(1+t)^x = 1 + \frac{x}{1} t + \frac{x^{(2)}}{2\,!} t^2 + \ldots$$

whence, by (18),

$$L_0 + tL_1 + t^2 L_2 + \ldots = \int_0^1 (1+t)^x dx = \frac{t}{\log_e(1+t)} \quad . \quad . \quad (24)$$

so that

$$t = \Big(L_0 + tL_1 + t^2 L_2 + \ldots\Big)\Big(t - \frac{t^2}{2} + \frac{t^3}{3} - \ldots\Big)$$

Performing the multiplication on the right-hand side and comparing the coefficients of corresponding powers of t on both sides, we find $L_v = 1$ and generally,

$$L_v = \frac{1}{2} L_{v-1} - \frac{1}{3} L_{v-2} + \frac{1}{4} L_{v-3} - \ldots + \frac{(-1)^{v+1}}{v+1} L_0 \quad . \quad . \quad (25)$$

The first few of these coefficients are

$$L_0=1,\ L_1=\frac{1}{2},\ L_2=-\frac{1}{12},\ L_3=\frac{1}{24},\ L_4=-\frac{19}{720},\ L_5=\frac{3}{160},$$

$$L_6=-\frac{863}{60480},\ L_7=\frac{275}{24192}.$$

It is easy to find simple limits for the coefficients L_v. Writing

$$t^{(v)}=-(t-2)(t-3)\ldots(t-v+1)\,.\,t(1-t)$$

and assuming $v>2$, we obtain from (18), by the Theorem of Mean Value,

$$L_v=-\frac{(\theta-2)(\theta-3)\ldots(\theta-v+1)}{v\,!}\int_0^1 t(1-t)\,dt\,(0<\theta<1)$$

or

$$L_v=-\frac{1}{6}\cdot\frac{(\theta-2)(\theta-3)\ldots(\theta-v+1)}{v\,!}\,(0<\theta<1) \quad . \quad . \quad (26)$$

It follows that L_v, for $v>0$, has the sign $(-1)^{v-1}$. Further, we obtain from (26)

$$|L_v|=\frac{1}{6}\cdot\frac{(2-\theta)(3-\theta)\ldots(v-1-\theta)}{v\,!}\,(0<\theta<1)$$

so that

$$\frac{1}{6v(v-1)}<|L_v|<\frac{1}{6v}\ (v>2) \quad . \quad . \quad . \quad . \quad (27)$$

L_v therefore tends to zero, although not particularly quickly, when v tends to infinity.

The formula (23), apart from the remainder-term, is of course well known.

Other formulas of a similar nature are obtained by inserting, in (14), for $f(t+r)$, various interpolation series with their remainder-terms. We shall not stop to examine them all, but confine ourselves to a particularly useful one.

If, in (13), we write $f(n+x)$ for $f(x)$ we obtain

$$\frac{f(n+x)+f(n-x)}{2}=\sum_{v=0}^{r}\frac{x^{[2v]}}{(2v)\,!}\delta^{2v}f(n)+\frac{x^{[2r+2]}}{(2r+2)\,!}f^{(2r+2)}(\xi)\,. \quad (28)$$

Writing

$$K_n=\frac{1}{n\,!}\int_{-\frac{1}{2}}^{\frac{1}{2}} t^{[n]}\,dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

whence

$$K_{2v+1}=0\,;\ K_{2v}=\frac{2}{(2v)\,!}\int_0^{\frac{1}{2}} t^2(t^2-1)(t^2-4)\ldots(t^2-\overline{v-1}^{\,2})\,dt \quad (30)$$

we obtain from (28)

$$\left.\begin{aligned}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(n+x)+f(n-x)}{2}\,dx \\ &= \sum_{v=0}^{r} \mathrm{K}_{2v}\delta^{2v} f(n) + \mathrm{K}_{2r+2} f^{(2r+2)}(\xi)\end{aligned}\right\} \quad . \quad . \quad (31)$$

and, summing from $n=0$ to $n=m-1$,

$$\int_{-\frac{1}{2}}^{m-\frac{1}{2}} f(x)\,dx = \sum_{v=0}^{m-1} f(v) + \sum_{v=1}^{r} \mathrm{K}_{2v}\left[\delta^{2v-1} f\left(m-\frac{1}{2}\right) - \delta^{2v-1} f\left(-\frac{1}{2}\right)\right]$$
$$+ m\mathrm{K}_{2r+2} f^{(2r+2)}(\xi) \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The remainder-term in (31) is, of course, obtained from (28) by the Theorem of Mean Value, noticing that $x^{[2r+2]}$ does not change its sign in the interval $\pm\frac{1}{2}$.

The formula (32) is also well known, apart from the remainder-term, and has the great advantage over (23) that all differences of even order have disappeared. The first few of the coefficients K_{2v}, calculated by (30), are

$$\mathrm{K}_2 = \frac{1}{24}, \; \mathrm{K}_4 = -\frac{17}{5760}, \; \mathrm{K}_6 = \frac{367}{967680}, \; \mathrm{K}_8 = -\frac{27859}{464486400}.$$

Applying the Theorem of Mean Value to (30) we find immediately

$$\mathrm{K}_{2v} = (-1)^{v-1} \frac{(1-\theta^2)(4-\theta^2)(9-\theta^2)\,\ldots\,(\overline{v-1}^2-\theta^2)}{12(2v)\,!} \left(0<\theta<\frac{1}{2}\right) \quad (33)$$

and, for $\theta=0$,

$$|\,\mathrm{K}_{2v}\,| \leqslant \frac{1}{12v^2\binom{2v}{v}}. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

It is seen by (33) that K_{2v} has the sign $(-1)^{v-1}$, and by (34) that the K_{2v} decrease much more quickly than the coefficients of (23) according to (27).

It is possible, by a similar method, to derive Lubbock's formula with the remainder-term of which no account was taken by Lubbock. We confine ourselves to stating the result which may be written

$$\left.\begin{aligned}\sum_{0}^{mk-1} f(v) &= m\sum_{v=0}^{k-1} f(vm) + \sum_{1}^{n}\Lambda_v\left[\Delta_m^{v-1} f(mk) - \Delta_m^{v-1} f(0)\right] + \mathrm{R} \\ \mathrm{R} &= km^{n+1}\Lambda_{n+1} f^{(n+1)}(\xi)\end{aligned}\right\} \quad (35)$$

Here, Δ_m is defined by $\Delta_m f(x) = f(x+m) - f(x)$, and the coefficients Λ_v are obtained from the well-known expression

$$\frac{x}{(1+x)^{\frac{1}{m}} - 1} = \Lambda_0 + x\Lambda_1 + x^2\Lambda_2 + \dots \quad . \quad . \quad (36)$$

For the sake of completeness we may also mention a few other formulas of summation or integration, first the well-known Euler-Maclaurin formula or

$$\left.\begin{aligned} &\sum_0^{h-1} f(v) = \int_0^h f(x)dx - \frac{f(h)-f(0)}{2} + \sum_{v=1}^{\mu-1} A_{2v}\Big[f^{(2v-1)}(x)\Big]_0^h + R \\ &R = hA_{2\mu} f^{(2\mu)}(\xi) \end{aligned}\right\} (37)$$

where $A_2 = \frac{1}{12}$, $A_4 = -\frac{1}{720}$, $A_6 = \frac{1}{30240}$, ...

The remainder-term of this formula has long been known. But there are two more related formulas where the remainder-term does not seem to have been given in a practical form. One is

$$\sum_0^{h-1} f(v) = \int_{-\frac{1}{2}}^{h-\frac{1}{2}} f(x)dx - \sum_{v=1}^{\mu-1} C_{2v}\Big[f^{(2v-1}(x)\Big]_{-\frac{1}{2}}^{h-\frac{1}{2}} - hC_{2\mu} f^{(2\mu)}(\xi) \quad (38)$$

with the coefficients

$$C_2 = \frac{1}{24}, \; C_4 = -\frac{7}{5760}, \; C_6 = \frac{31}{967680}, \dots$$

and generally

$$C_{2v} = \left(1 - \frac{1}{2^{2v-1}}\right)A_{2v} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The other formula is

$$\left.\begin{aligned} &\sum_0^{mk-1} f(v) = m\sum_{v=0}^{k-1} f(vm) + \frac{m-1}{2}\Big[f(x)\Big]_0^{mk} \\ &\qquad\qquad - \sum_{v=1}^{\mu-1} A_{2v}(m^{2v}-1)\Big[f^{(2v-1)}(x)\Big]_0^{mk} + R \\ &R = -mkA_{2\mu}(m^{2\mu}-1)f^{(2\mu)}(\xi) \end{aligned}\right\} . \quad . \quad (40)$$

The proofs I possess of the remainder-terms in (38) and (40) are quite elementary, but require some previous knowledge of the properties of Bernouilli's polynomials, and would claim too much space here.

The advantage of (38) over the usual form of Euler-Maclaurin's formula is not very great; but (40) compares favourably with Lubbock's formula, as the terms of even order have disappeared.

There is another class of formulas of approximate integration where the integral is represented by a linear function of a certain number of equidistant values of the integrand. I have dealt with this class in two recent papers* and shall not go into the subject here.

It may finally be pointed out, that *repeated* integrals can always be reduced to single ones. For it may be proved by induction, that

$$\int_0^t\int_0^t\dots\int_0^t f(t)\,dt^{n+1}=\frac{1}{n!}\sum_0^n(-1)^v\binom{n}{v}t^{n-v}\int_0^t t^v f(t)\,dt$$

whence

$$\int_0^1\int_0^t\dots\int_0^t f(t)\,dt^{n+1}=\frac{1}{n!}\int_0^1\sum_0^n(-1)^v\binom{n}{v}t^v f(t)\,dt$$

$$=\frac{1}{n!}\int_0^1(1-t)^n f(t)\,dt.$$

* Skandinavisk Aktuarietidskrift 1921 and 1922: (1) On the Remainder Form of certain formulas of Mechanical Quadrature. (2) On Numerical Integration of Differential Equations.

LEGAL NOTES.

By Robert Allen Bateman, B.Sc. (Econ.), *Barrister-at-Law.*

Ballantyne's Trustees v. Scottish Amicable Life Assurance Society, 1921 W.C. & Ins. Rep. 262.

Where under the suicide clause in a policy all moneys paid under the policy shall belong to the company except in the case of a party onerously interested, is an agreement between a husband and wife, providing for separation and assigning the policy to trustees to secure certain provisions, onerous so as to entitle the wife to the policy moneys?

Yes, said Lord Hunter in a case where Ballantyne had effected a policy on 21 January 1920, which contained a clause that "in case any life assured by this policy shall die by suicide within one year from the date hereof, this policy shall then become void,

and all moneys paid under it shall belong to the society, except as regards the interest of any party onerously interested therein other than the life assured. Mrs. Ballantyne had apparently had good reason to suspect her husband's fidelity and as a consequence an agreement was entered into on 6 February 1920, under which certain pecuniary provisions were made in favour of the wife and children. To secure these provisions Ballantyne assigned to trustees the policy of assurance among other things. The agreement had a provision to the effect that 'in regard to the proceeds of the said policy or policies of assurance, together with any bonus or other additions thereon, I hereby provide and direct that the trustees shall on my death, should my wife survive me, pay these over to my said wife as her own absolute property, and that whether or not the provisions of this agreement which refer to our living apart come into operation; declaring that the sum derived therefrom shall be treated as a payment to account of my wife's legal share of my estate on intestacy.'" On 5 March 1920, Mr. Ballantyne died by suicide. In giving judgment Lord Hunter said: "The agreement was, "I think, onerous—at all events so far as the spouses were "concerned. Mrs. Ballantyne condoned the marital offence "of her husband and refrained from prosecuting an action of "divorce, which, if successful, would have given her pecuniary "rights in her husband's estate, in return for the provisions "made for her and her children in the minute of agreement and "the assignation made in security of these provisions. Counsel "for the defenders was right in maintaining that the provisions "are to some extent testamentary, and the question whether "in the event which has occurred the trustees are in the position "of representatives of Mr. Ballantyne, at all events to the "extent of two-thirds of the money, is a question of considerable "difficulty. It is admitted that Mr. Ballantyne died intestate, "leaving a substantial estate, and the effect of the clause in the "agreement which is quoted above is that the estate of "Mr. Ballantyne largely benefits from the policy money. The "defenders say this is contrary to the terms of the policy.

"For the pursuers a strong argument was presented to the "effect that onerosity fell to be determined at the time of the "assignation, and that in the event which has occurred—that "is, the husband's predecease of the wife—there is a direct "gift of the policy money to the wife, for whom the trustees "hold."

After referring to a number of decided cases where policies containing a similar suicide clause had been considered and the estate of the persons who had committed suicide had benefited as a result of the decisions, Lord Hunter continued: "All the "cases to which I have referred were mortgage cases, but it "appears to me that the reasoning in the opinions quoted is "equally applicable where the claim is made in respect of an "absolute assignation. In the event which has occurred the "trustees are bound to assign the whole of the policy money to "Mrs. Ballantyne. If I am right in the view that the agreement "with her husband was onerous, she would have been entitled "to the whole of the £5,000 if her husband had left no additional "estate. I do not think her right is any different because she "has to ascribe what she receives under the policy to *jus relictæ*. "The estate of the deceased is benefited, but as Mrs. Ballantyne "is onerously interested in the whole of the policy, I do not "think that that circumstance prevents effect being given to "her claim or to that of the trustees who take on her behalf."

Poulter & Waldron v. Fitzimon & Lloyd (1922), 1 I.R. 20.

Where a policy is settled on a person's issue with remainder to the settlor on failure of issue and subsequently mortgaged, can the policy moneys be paid to the mortgagee on the death of the settlor (and mortgagor) provided the mortgagee effects a policy payable on the birth of issue?

Yes, said Powell, J., in a case which raised not only this point but also the further point whether there is a presumption of law as to the age when the possibility of child-birth becomes extinct. The facts of the case were that one Fitzimon in 1901 settled a policy of assurance on his own life for £2,000 on trust in equal moieties for the issue of two daughters with ultimate remainder in both moieties in default of issue to himself. In 1903 Fitzimon mortgaged the policy to one Rowe who, in 1909 assigned his interest to one Lloyd. Later in 1909 Fitzimon died and after an action brought to administer the trusts of the settlement of 1901 the balance of the policy moneys, after certain payments, was lodged in Court. In 1915 Lloyd died.

In 1916 one daughter married but up to the time of the present action, when she was 45 years of age, there was no issue. The other daughter remained unmarried. The present action was by way of summons asking that the money in Court should be paid out to Lloyd's executors on their effecting and lodging with the Court two policies of assurance at single premiums with the Scottish Union and National Insurance Company, to be paid on birth of issue as to a moiety for the issue of each daughter. In the course of argument counsel referred to the fact that there was no presumption of law as to the age when the possibility of child-birth becomes extinct and quoted the judgment of Lord Lindley in *In re* Hocking (1898) 2 Ch. 567, where he said: "There, in *In re* Lowman (1895) 2 Ch. 348, the Court deprived "no living person of a possible interest; nor, so far as I am "aware, has it ever done so. If property is given to A, in the "event of B having no children, can A claim that property "before the death of B? My answer is no, neither at law nor "in equity, unless B's possible child is the only person who can "deprive A of the property"; which would be the case here. In making an order in the terms of the summons, Powell, J., said: "I do not decide that at the age of fifty-one or forty-five "the possibility of child-birth is presumed to be extinct. I "make the order in the special circumstances of the case, and "because I consider that provision is thereby made for any "issue who might become entitled under the settlement."

Horne v. Poland and Others, 38 T.L.R. 357.

Where an alien takes out a policy of insurance, may the fact that he is an alien be a circumstance the non-disclosure of which to the insurers will avoid the policy?

Yes, said Lush, J., in a burglary insurance case, although the same principle would no doubt be applied in other branches where similar facts were in question. In the present case the plaintiff was born in Roumania and his real name was Euda Gedale. He had been in this country over twenty years but had not been naturalized. He changed his name soon after arriving in this country. In the course of his judgment Lush, J., said:

"He (the plaintiff) registered himself under the Act for the "registration of aliens, but when he effected the policy in "question, and when he was insured, as he was before by the "Ocean Company, he described himself as Harry Horne and "did not state that he was a Roumanian or that he had lived in "Roumania for many years. It was not alleged by the "underwriters that he acted fraudulently in that respect. What "was said was that he did not disclose facts which were material "to the underwriters to know, and that therefore the policy "was invalidated. Evidence . . . given by underwriters amounted "to this; that underwriters would not accept a proposal of an "alien for insurance as satisfactory and would not accept the "risk. . . . . even apart from evidence he could not feel any "doubt that, in the circumstances of the case, the fact was "material. . . . . He could not agree with the view which "was pressed upon him . . . . that the mere fact that a person "was a foreigner and not a British subject was in all cases a "material fact, so that the non-disclosure of it invalidated the "policy. One could easily think of cases in which it could not "affect the mind of a reasonable underwriter. . . . Each case "must depend on its own circumstances. . . .

"The question of law was usually stated as in Lord Justice "Moulton's judgment in Joel's case (1908) 2 K.B. 863. Counsel "for the plaintiff had contended that it must also be shown "that the assured was under a duty to disclose the fact, and "that the plaintiff could not be assumed to know that people "of his nationality were not as useful and trustworthy, and "good for insurance, as Englishmen, and that therefore he had "no knowledge of any circumstances from which, as a reasonable "basis, he should have treated the facts in question as material. "In the judgment of Lord Justice Moulton in Joel's case, the "Lord Justice was only saying . . . . that if an ordinary person, "taking reasonable care and using reasonable judgment, would "not know all the facts—if, for example, he would not know "that a particular symptom indicated the presence of a serious "malady—he was under no duty to state that he had no "symptom, because all the material facts would not be within "his knowledge. If they were within his knowledge, or if they "would be within the knowledge of an ordinary reasonable "person, then, the Lord Justice said, he must disclose them. "It was immaterial that the plaintiff would not take the view "that his countrymen were not as careful and trustworthy as

"Englishmen. That was not the real question. He knew all "the facts, all the circumstances. If the ordinary reasonable "person would know that underwriters would naturally be "influenced, in deciding whether to accept the risk and what "premiums to charge, by those circumstances, the fact that "they were kept in ignorance of them, and indeed were misled, "was fatal to the plaintiff's claim. . . . The case was a hard "one. A great many aliens were residing and carrying on "business in this country, and probably many of them were "insured by policies which were unenforceable, although quite "honestly effected. . . . It would seem more just if underwriters "had to ask as to the nationality of the person they proposed "to insure if they attached importance to it and were going to "rely on it."

REVIEW.

The Use of Death Rates as a Measure of Hygienic Conditions. By JOHN BROWNLEE, M.D., D.SC.

[Medical Research Council. Special Report Series, No. 60,]

[Pp. 80. London: H.M. Stationery Office, 1922. Price 3*s*. net.]

THE death rate is not a function which enters into actuarial calculation, and the interest of members of the Institute in this report is likely to be directed mainly to the mathematical treatment of the subject, and to the short methods indicated of forming life tables—in this volume for the purpose of obtaining life table death rates and expectations of life. These methods illustrate how, on hypothetical bases of population, very close approximations to death rates or, by taking the reciprocal, expectations of life can be obtained for any particular population which may be under investigation.

Part I deals with death rates in general, distinguishing between crude death rates (Section II), standardized death rates (Section III), and life table death rates (Section IV). The defects of crude death rates hardly need comment here. As regards standardized death rates both direct and indirect methods are illustrated. By the direct method the standard population is arranged in age groups, the numbers in each group being multiplied by the death rate for the corresponding age of the selected population, the total number of

deaths thus obtained being divided by the standard population and brought to a rate per thousand. By the indirect method the death rates for the respective age groups of the standard population are multiplied into the population for the corresponding age group in the selected population, the number of deaths expected being divided by the selected population and brought to a rate per thousand; the ratio between the death rate in the standard population and that brought out at this stage being used as a correcting factor to be multiplied into the crude death rate for the selected population. In passing it may be pointed out that either method should be used with caution. Generally the indirect method would seem preferable where there are relatively few numbers of the selected population in individual age groups, as the direct method gives abnormalities in particular groups undue weight. If however the indirect method were used to obtain a standardized death rate amongst a body such as railway engine drivers as set out in Table II, where there is an element of withdrawal at the older ages, the indirect method would obviously produce a figure understating the true position.

The life table death rate is accepted as the most adequate measure for purposes of comparison, though its defects by reason of the assumptions of a stationary population and a constant environment are recognized. The term "life table death rate" is strictly defined as the ratio of the number of deaths of persons *above any defined age* to the number living above that age in a stationary population. The outstanding advantage possessed by the life table death rate is, of course, that by taking its reciprocal the expectation of life is obtained.

After thus dealing with the question of death rates reference is made to methods suggested by Farr and Pearson (Section V) and the report passes to mortality at different ages in various districts, in comparison with that of healthy districts, by means of ratios (Section VI). Stress is laid on the ratios but the actual differences in the rates should not be forgotten, especially at the young ages.

Section VII deals with the connection between life table and standardized death rates, but is not important, as a later method is given (see Table XIV) of obtaining the life table death rate direct from the death rates of each age by reference to a standard population at each age decreasing in arithmetical progression. The result is not very different from standardizing direct to the assumed stationary population of a life table and the results bring out close approximations to the expectation of life. On the question of the construction of abridged mortality tables reference may be made to the review of Part II of the Supplement to the 75th Annual Report of the Registrar-General in *J.I.A.*, vol. lii, p. 393.

Sections are devoted to various diseases but it is general method

rather than particular application that is of interest from the actuarial point of view, and detailed comment is unnecessary here.

Generally speaking the report has been well prepared, the various questions involved being carefully and fairly considered. The arrangement, too, for a report of this kind is good, enabling students to select and apply themselves to those sections in which they may be specially interested.

S. J. P.

④ JOURNAL

OF THE

INSTITUTE OF ACTUARIES.

Austrian National Life Tables. *By* George William Richmond, F.I.A.

[Submitted to the Institute, 27 March 1922.]

The problems of vital statistics are of the utmost importance as affecting in many ways the welfare of the people. In connection with the study of such problems the value of reliable life tables applicable to the general population is well recognized. The preparation of national life tables presents various difficulties, however, if the results are to command respect for their reliability. It is therefore in the hope that British actuaries will be interested to see how these difficulties have been met in another country that the following notes have been prepared.

Following the world upheaval of the last seven years, it is quite possible that deep rooted changes in the vital conditions of our own and other countries may set in, which will call for the closest investigation. In conducting such an investigation, the fullest knowledge of available methods adopted, whether here or abroad, cannot but be advantageous.

It is accordingly proposed to describe briefly the preparation of the Austrian Population Life Tables, 1901–1910, the source of information being "Die Ergebnisse der Volkszählung vom 31 Dezember 1910 in Osterreich" Vol. I., Part 4, "Osterreichische Sterbetafeln" (Karl Gerold's Sohn, Vienna. 1917).

A feature of interest is that, as will appear, the work presents an example of close co-operation between Government officials and representatives of insurance companies, while a Corresponding Member of the Institute of Actuaries, Professor Dr. Ernst

Blaschke, Court Councillor, played a most prominent part throughout. It will be a real privilege to place some results of his labours before a wider audience.

The International Actuarial Congresses gave a definite impetus in the direction of harmonizing as far as possible the statistical researches conducted in different countries, and the advantage of this is evident. It is by comparison and analysis of differences and resemblances that knowledge is gained which may enable the lot of man to be ameliorated. No differences of opinion should hinder science in the pursuit of such an aim. On the other hand, a rigid uniformity of practice in all probability would defeat equally the progress which should be aimed at. Thus in the meantime a full acquaintance with work done elsewhere would appear to be the most useful.

Censuses and National Life Tables in Austria.

A census law has existed in Austria since 1869, and various attempts have been made to compile mortality tables from the results of the censuses which, however, have not come into general use. When national life tables have been required in connection with insurance business, it has been customary in general to make use of German mortality tables, where no statistics bearing immediately on the subject in view were available, *e.g.*, for the Accident and Employees Insurance Law. For Industrial insurance by private insurance companies, in four cases a Prussian, and in three cases a German population mortality table was employed. Austrian population tables have not been used.

Viktor Kitz first published an Austrian mortality experience from the results of the census of the year 1869 and the birth and death registers of the years 1865–1875, and then he built up a mortality table based on the censuses of the years 1869 and 1880 and the birth and death registers of the years 1870 to 1880. Two tables were derived from the results of the census of the year 1900; one with the help of the death registers of the years 1895–1900 for the probabilities of death in quinquennial age groups, the other for the mortality of the year 1901.

In connection with the preparation of the census of 31 December 1910, the Insurance Section in the Ministry of the Interior urged strongly, in advance, that the results of the enumeration should be arranged to provide mortality tables serviceable for their purposes. The Central Statistical

Commission felt itself the more impelled to fall in with this, since it was known that Austrian Insurance Institutions made use in case of need of German mortality tables, so far as they did not possess special material. The matter was discussed on the 24 February 1910 in a sitting of the special Committee of the Central Statistical Commission for the preparation of the Census, and took shape when, with the memorandum of the late President of the Central Statistical Commission, Dr. Robert Meyer, of 3 January 1914, a plan of work was forwarded to the then President of the Department for Private Insurance in the Ministry of the Interior, Dr. Ernst Blaschke, to test and report upon. Since the construction of actuarial tables (Annuity Tables), which was afterwards suggested, went beyond the original plan, the Austro-Hungarian Association of Private Insurance Institutions, in reply to an enquiry by the Central Statistical Commission, declared itself, in February 1914, ready to provide personal assistance, and to take over a part of the editing of the special number devoted to mortality, on payment of a suitable contribution.

The far-advanced labours suffered, so to say, in the twelfth hour, a serious delay, since several of the assistants who were especially entrusted with the work of calculation were called up for military service. Not until the spring of 1917 did the possibility offer itself to complete the work, when, on the suggestion of Councillor Blaschke, the Austro-Hungarian Association of Private Insurance Institutions again demonstrated its warm interest in furthering this matter by taking over the expenses for securing specially capable assistants who did not belong to the Department.

Austrian Mortality Tables from the Census of 31 *December* 1910.

Vol. I, Part 4 of the "Ergebnisse" gives various mortality tables derived from the statistics of the Austrian population for the decennium 1901–1910 inclusive. After a preface, there is an introduction which shows how the tables have been built up: then there are given auxiliary tables to test the general accuracy of the whole. The next section of the introduction is devoted to the question of the mortality of children in Austria, which is followed by another section giving a comparison of the results of Tables I to VII* with those of other mortality experiences.

* Tables of the above-mentioned volume of "Ergebnisse" quoted in roman figures.

The tables themselves are as follows :

Tables I *to* III.—Observations for the calculation of Austrian Mortality Tables for the decennium 1901–1910, derived from the results of the Censuses 1900 and 1910, and the Death and Birth Registers of the years 1901 to 1911.

(1) The number of the living in years of age during the decennium 1901–1910.

(2) Deaths in the decennium 1901–1911.

(3) Population and migration according to generations.

Tables IV *and* V.—*Austrian Mortality Tables.*

(4) General Austrian mortality Tables for the decennium 1901–1910, the quinquennium 1901–1905 and 1906–1910 (graduated).

Tables VI.—*Actuarial Tables :* (*a* to *f*).

(6) Elementary values of a general Austrian Mortality Table for the quinquennium 1906–1910.

(*a*) Male sex (ungraduated).
(*b*) ,, (graduated according to Gompertz-Makeham).
(*c*) ,, (graduated according to Bruns).
(*d*) Female sex (ungraduated).
(*e*) ,, (graduated according to Gompertz-Makeham).
(*f*) ,, (graduated according to Bruns).

Table VII.—Rates of Mortality according to geographical areas, based upon the living according to the Census of 1910 and the deaths in the years 1910 and 1911.

In their preface to the above-mentioned volume, Dr. Karl Ritter Eisler von Eisenhort, Vice-President of the Imperial and Royal Central Statistical Commission, and Dr. Wilhelm Hecke, Government Councillor, observe as follows :

"An explanation of the methods of calculation employed, "with a general theoretical demonstration which is reprinted "here as an appendix, appeared previously, in the December "number of the 1916 Annual of the Statistical Monthly Journal. "The following introduction to this number has been prepared "by the author of the appendix, Professor Dr. Ernst Blaschke, "corresponding member of the Central Statistical Commission. "In addition to such far reaching assistance, which materially "increases the value of the following publication, this noted "expert has devoted himself also to supervising and testing the "heaviest part of the calculations in an equally unselfish manner, "as he always strove to facilitate the completion of the work.

"To him for his many-sided and successful co-operation warmest "thanks are here expressed. Likewise Professor Emanuel "Czuber has taken special interest in the work. On examination "of the appendix already printed, he discovered some mistakes, "which are brought together at the end of the introduction. "Best thanks are due to him also for his co-operation."

Introduction.

The Derivation of the Tables.

In order that the mortality tables should be practically applicable, the programme of work was so arranged that death tables should be obtainable, giving the rates of mortality for the average of several years according to sex and geographical areas. Moreover, the material was to be published in such a form that it could serve as a basis for further research. To this end two questions were prominent: one statistical and one actuarial.

The mortality table was to be based on the two censuses of the years 1900 and 1910 as well as the registers of births and deaths of the intervening years. It was consequently desirable first of all to construct a mortality table for the whole period from 1901 to 1910. Table IV contains the relative rates of mortality and the expectation of life at each age. In view, however, of the circumstances that the rates of mortality throughout the ten years in question showed a remarkable downward tendency, these figures alone were inadequate from a scientific standpoint. The material was, therefore, in advance, so arranged that the rates of mortality mentioned should be given also separately for the period 1901 to 1905, and for 1906 to 1910. The latter figures seemed also to give a suitable junction to the results for the quinquennium 1896 to 1900, in which this measure of mortality was likewise developed for a five year period. It seemed, however, undesirable even to stop here, but so to arrange the publication of the material as not to exclude the possibility of an investigation of the mortality likewise for smaller periods of time, going even so far as for one year only. Hence the observations, the aggregate of living and of the deaths, were developed for each individual year between 1901 up to 1910, and were printed in Tables I and II. Thus, in effect, the material of exposed to risk and deaths is given completely

(*a*) for each sex
(*b*) for every year of age
(*c*) for every calendar year 1901–1910 inclusive

affording opportunities for the construction of separate mortality tables for each sex and each calendar year. Further study may show certain relationships of value between the rates for sex and sex, and some approximate formulas may be obtainable for generalizing the secular changes in the rate of mortality, though this may not be easy in consideration of the fact that the rate of change (improvement, in the present instance) varies at the different stages of life and for each sex. But in any event the more complete supply of facts provides the opportunity for a broader view and closer study of the subject. At the least the variation in the rate of mortality from calendar year to calendar year can be examined and compared with the range allowed by the theory of probabilities. Thus the reliability of studies based on the latter theory may be retested in the light of this practical experience.

Monetary Functions.

In order to secure the greatest possible utility for all purposes, monetary functions were calculated at 3, $3\frac{1}{2}$ and 4 per-cent

(*a*) for each sex
(*b*) for (1) the ungraduated table
(2) the experience graduated by the formula of Gompertz-Makeham
(3) the experience graduated by the formula of Bruns
(*c*) for the quinquennium 1906–1910, as representing the most recent and reliable experience founded upon an adequate basis, the functions calculated for these numerous tables are the commutation values.

Students referring to the original must bear in mind the difference in notation. It goes beyond the purpose of the present notes to deal further with these extensive tables of monetary functions.

Geographical Variations in Mortality.

The area of the late Austrian Empire was inhabited by a great variety of races and peoples, with extraordinary differences in economic development. Probably any analysis by races is out of the question, but the following summary shows the rates of mortality in the then great provinces, Vienna, Sudete and the

Carpathians, for the period of 1870/80 and for 1910 (sheet 16), demonstrating the improvement in vitality that has taken place, particularly at the younger ages. The 1910 figures were based on the Census of that year and on the deaths within a year. The rates of mortality are given separately for each of the first five years of life, then for the age group five to nine, and then from age ten, for each decennial age group. Similar summaries of the rates of mortality in other provincial areas were prepared on the basis of the 1910 census, but corresponding figures for the period 1870/80 were not available.

With reference to the requirements above mentioned for carrying out the mortality investigation and arranging the actual tables, the method of preparation of the statistical material and for arranging the latter tables, was fairly precisely laid down, and only a very brief explanation of it is given here.

According to the plan of work, it was arranged to construct mortality tables, that is tables showing rates of mortality and the successive numbers of the living with the resulting numbers of deaths, and the expectation of life. This is possible, when it can be ascertained how many persons complete each calendar year and how many persons die within a year out of this aggregate. This was directly ascertainable from the experience. The aggregates of the deaths are given in Table II, the aggregate of the living in Table I, for each calendar year, 1901–1910. The former aggregates were obtainable directly from the reports of Austrian statistics regarding the movement of the population. They are given there with an amount of detail that exceeds the immediate object, namely, according to year of birth, age and year of death, for the distribution of deaths according to year of birth and age would suffice. The figures of Table II are consequently numbers that have already been prepared, the deaths of the like age and generation being combined from two successively following years of death.

The aggregates of the living in Table I, the number of persons who in one of the ten calendar years completed a fixed age, are derived by an elaborate process of calculation. They were developed from the statements of the population according to the Censuses of 1900 and 1910 as well as the number of births in the calendar years 1901 to 1910, which statements are produced fully in Table III, and finally from the above-mentioned aggregates contained in the report of Austrian Statistics of deaths. For example, the aggregates of living of those persons

who in the census year 1910 completed a fixed age are ascertained, if to the figures of the population of a fixed year of birth, there are added the deaths due to the same year of birth of that age on the census day and dying in 1910.

The survivors in the preceding calendar years for an age one unit lower which are obtained, apart from the correction for migration, by adding to the column of living as formed the deaths of the same generation and of the preceding age at death from Table II, the survivors in the second preceding calendar year two years of age less again, apart from the correction of migration, by adding to the living as above, formed the deaths of the like generation and second preceding age at death from Table II.

The above sketched scheme of calculation in order to obtain the figures of Tables I and II was systematically carried through, as shown in the following example of part of the schedules used.

This procedure deviates considerably from that of Johannes Rahts (described in vol. 200 of the Statistics of the German Empire), being much more complicated. By the method of Rahts the aggregates of living according to the census, that is, "the aggregates of those surviving any age" are obtained directly by adding to the aggregate of the living at the next age the sum of the deaths at the given age and of the balance of migrations and of those surviving at both ends of the period. The application of the method of Rahts leads, however, only to the aggregate of the living for the entire period (1901–1910), but not to the intermediate aggregates for each of the calendar years 1901–1902 . . . . . to 1910, that is, the living from year to year. Thereby is lost the possibility of extending the investigation to individual calendar years.

The result of the schedule from columns 8, 15, 21, &c., gives the numbers of Tables I, the result of columns 6, 12, 19, &c., gives the numbers of Table II. In general, the whole of the working of the Schedule is evident: only the migration corrections of columns 13, &c., require some further explanation.

Treatment of Migrations.

Had no migration taken place among the population during the whole period under observation, 1901–1910, then from each year of birth one would pass from the numbers of the living at the Census of 1910 to those at the Census of 1900 by adding successively the deaths during the calendar years 1910, 1909, &c.

Schedule for Calculating the Aggregate of the Living and Deaths in Tables I *and* II.

Year of Birth	Numbers Living 31 Dec. 1910	Deaths in Adjacent Years of Birth				There were thus in 1910 of this Age at least		Deaths in Adjacent Years of Birth			
		Age	1911	1910	Together	Age	Persons	Age	1910	1909	Together
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1910	397,221	1	26,824	72,063	98,887	0	469,284	...	...	...	...
1909	337,993	2	7,303	10,710	18,018	1	348,703	1	25,043	78,633	103,876
1908	...	...	...	...	...	...	...	...	...	...	...

Continuation of above Table.

Year of Birth	Average Annual Migration for the adjacent Year of Birth Immigration (−) Emigration (+)	There were thus in the Year 1909 of this Age at least (Cols. 8+12±13)		Deaths in Adjacent Years of Birth				There were thus in the Year 1908 of this Age at least (Cols. 15+19±13)	
		Years	Persons	Age	1909	1908	Together	Years	Persons
(1)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)
1910	...	...	...	...	...	...	...	...	...
1909	+21,105	0	473,684	...	...	...	...	...	...
1908	+ 2,391	1	373,585	1	30,530	76,817	107,437	0	483,323

in accordance with the Schedule but without correction of migration, &c., but since migration did take place, there remains a difference between "expected" and "actual" numbers of the people at each year of birth in 1900, which will be positive if more people have emigrated than have immigrated, and negative in the opposite case. The calculation was carried through for every year of birth, and the question answered, how great the difference due to migration in every calendar year would have been, had the migration been distributed uniformly throughout the decennium (supposing the total migration of each generation in general had been divided into 10 equal parts). The result of this calculation is shown (in addition to the numbers shown in the Census in columns 2 and 3) in column 4 of Table III, the numbers of which moreover as a balance of migration were transferred into the columns mentioned as an equal amount within each generation.

Treatment of Inaccuracies in Age.

A further observation appears to be desirable regarding this Schedule. The Census results and draft records were not recorded in the Schedule unaltered. As is generally known, there is a tendency to a heaping up around the quinquennial ages. The numbers consequently were redistributed by a smoothing process described later. By these preparatory measures the aggregates of living and dead were formed for each calendar year, and it is possible with their help to calculate the rate of mortality for each age and calendar year. If, for example, we wish to know the rate of mortality for age 40 for the males of the calendar year 1901, then it is only necessary to take the ratio between the corresponding values of Tables I and II, namely, 1,167 divided by 148,934 = ·0112. All other calculating operations which were performed with these figures served to render them more useful for practical purposes. If, for example, we examine the rate of mortality derived from the observations shown in Table IV they will be found to show unjustifiable irregularities (fluctuations): thus, in spite of the generally increasing tendency of the rate of mortality, there is a decrease from age 27 to age 28, &c. These were removed by the method of graduation.

Graduation.

The methods whereby a set of finally adjusted tables was

arrived at will be dealt with more fully later, but it may be mentioned here that all four mortality tables, namely:

(*a*) Males and females.

(*b*) One of each for the periods 1901–1905 and 1906–1910, were graduated first of all by the formula of Gompertz-Makeham, with the idea of producing tables of sufficient accuracy for practical purposes, while retaining the well-known advantages of tables following that formula. But since that formula removes not only accidental irregularities but also those that may be characteristic of the particular experience at least, if not of mortality curves in general, the differences between the figures of the graduated and ungraduated tables were subjected to further processes. Such further processes were the application of a graphic method to all four tables, and further of the Bruns method to the Male and Female Tables, 1906–1910. It should be especially noted that these further processes were applied to the differences between the ungraduated but smoothed tables and the Gompertz-Makeham graduations: that the entire tables were not regraduated.

The Mortality of Children in Austria.

The investigations into the mortality of the Austrian population during the years 1906–1910 were finally completed by a special investigation into the mortality of children during the first year of age. For this there were available on the one hand the birth registers and on the other the cases of death of children during the first, second and third month, in the second, third and fourth quarters of the first year of life, separately by generations for the year of death and for the year preceding, contained in the reports on the movements in the population.

Thus the death totals requisite to form the rates of mortality for age periods less than a year could be derived by addition of the deaths for that section of age and two successive calendar years, and the totals for the living from these new death totals by subtraction from the births. If for example m^k denotes the deaths of children aged 0 to 1 month in the calendar year k and m^{k+1} the deaths of the same age in calendar year $k+1$, then out of the births g^k of the calendar year k have resulted in the first month of life m^k+m^{k+1}, leaving at the beginning of the second month of life $g^k-(m^k+m^{k+1})$, and the rate of mortality in the first month of life is $m^k+m^{k+1}\div g^k$. Analogously we obtain as the rate of mortality in the second month of life

$m_1^k + m_1^{k+1} \div g^k - m^k - m^{k+1}$ where m_1^k and m_1^{k+1} represent the deaths from the first to the second month of life, &c.

In this way were prepared the following summaries separated for generations.

The tables call for the following remarks :

1. Since migrations are not allowed for in Table IV the living aged 1 do not agree exactly with those of Table IV. We have for males 78,168 instead of 78,018 in Table IV, and for females 81,554 instead of 81,688 living. Consequently the differences between Table IV and IVA were distributed according to the number of deaths in each section. The result forms Table VA.

2. The average probabilities of survivorship were calculated on the usual hypothesis of a uniform distribution during the section of age under observation. There resulted for example this value for age 0 from the formula

$$\left\{\frac{1}{24}(d_{\frac{0}{12}} + 3d_{\frac{1}{12}} + 5d_{\frac{2}{12}} + 9d_{\frac{3}{12}} + 15d_{\frac{6}{12}} + 21d_{\frac{9}{12}}) + 0{\cdot}5l_1 + \sum_1^{\infty} l_i\right\} \div l_0$$

where $d_{\frac{0}{12}}$ = the deaths in the first month of age,

$d_{\frac{1}{12}}$ = the deaths in the second month, and so on, and

l = the living.

3. In comparison with the German Empire are obtained the following values for the rates of mortality multiplied by 100.

Rates of Infant Mortality × 100.

Months of Age	Males			Females		
	Austria	Germany		Austria	Germany	
		1890-1900	1900-1910		1890-1900	1900-1910
0–1	8·764	7·372	6·539	6·830	5·782	5·129
1–2	2·710	3·111	2·590	2·232	2·583	2·109
2–3	2·200	2·711	2·280	1·760	2·247	1·815
4–6	4·570	5·890	4·938	3·692	4·957	4·105
7–9	3·277	3·965	3·326	2·801	3·450	2·910
10–12	2·630	2·913	2·436	2·482	2·637	2·261

Table Ia.

Number of the Living at the undernoted Years of Birth.

Year of Birth	Number of Persons at the beginning of the undernoted Months of Age					
	1	2	3	4	7	10
	Males					
1906	492,344	448,481	435,776	425,793	405,484	391,944
1907	483,792	440,644	428,830	419,369	399,926	386,588
1908	484,519	441,922	429,974	420,566	401,540	388,398
1909	484,236	442,564	430,682	421,523	402,755	389,757
1910	474,420	435,129	424,080	415,085	397,282	385,083
Total	2,419,311	2,208,740	2,149,342	2,102,336	2,006,987	1,941,770
	Females					
1906	468,914	435,863	425,610	417,474	401,034	389,412
1907	458,377	426,057	416,381	408,997	393,588	382,204
1908	456,856	425,247	415,639	408,428	393,062	381,765
1909	457,003	425,725	416,386	409,140	394,190	383,218
1910	449,125	419,686	410,742	403,742	389,650	379,190
Total	2,290,275	2,132,578	2,084,758	2,047,781	1,971,524	1,915,789

Table IIa.

Deaths according to Years of Birth.

Year of Birth	Number of Deaths during the undernoted Months of Age					
	1	2	3	4–6	7–9	10–12
	Males					
1906	43,863	12,705	9,983	20,309	13,540	10,556
1907	43,148	11,814	9,461	19,443	13,338	9,928
1908	42,597	11,948	9,408	19,026	13,142	11,226
1909	41,672	11,882	9,159	18,768	12,998	9,397
1910	39,291	11,049	8,995	17,803	12,199	9,550
Total	210,571	59,398	47,006	95,349	65,217	50,657
	Females					
1906	33,051	10,253	8,136	16,440	11,622	9,963
1907	32,320	9,676	7,384	16,409	11,384	9,290
1908	31,609	9,608	7,211	15,366	11,297	10,183
1909	31,278	9,339	7,246	14,950	10,972	8,686
1910	29,439	8,944	7,000	14,092	10,460	8,873
Total	157,697	47,820	36,977	76,257	55,735	46,995

Table IVa.

General Austrian Mortality Table for the Quinquennium 1906–10 (*ungraduated*).

Age	Males				Females			
Months	Probability of Death	Living at Beginning	Deaths	Expectation of Life at Beginning	Probability of Death	Living at Beginning	Deaths	Expectation of Life at Beginning
0–1	0·08704	100,000	8,704	40·644	0·06886	100,000	6,885	42·838
1–2	2689	91,296	2,455	44·514	2242	93,115	2,089	46·000
2–3	2187	88,841	1,943	45·740	1774	91,026	1,614	47·056
4–6	4535	86,898	3,941	46·748	3724	89,412	3,330	47·900
7–9	3249	82,957	2,696	48·963	2827	86,082	2,433	49·740
10–12	2609	80,261	2,093	50·588	2504	83,649	2,095	51·165

Table Va.

General Austrian Mortality Table for the Quinquennium 1906–10 (*graduated*).

Age	Males				Females			
Months	Probability of Death	Living at Beginning	Deaths	Expectation of Life at Beginning	Probability of Death	Living at Beginning	Deaths	Expectation of Life at Beginning
0–1	0·08764	100,000	8,764	40·644	0·06830	100,000	6,835	42·837
1–2	2710	91,236	2,472	44·544	2232	93,165	2,074	45·977
2–3	2200	88,764	1,956	45·781	1760	91,091	1,602	47·021
4–6	4570	86,808	3,968	46·806	3692	89,489	3,306	47·860
7–9	3277	82,840	2,715	49·033	2804	86,183	2,415	49·680
10–12	2630	80,125	2,107	50·673	2482	83,768	2,080	51·096

Comparisons with other Tables.

Undernoted is a comparison between the annuity values of the Population Table, Males, with those for assured lives according to the Table AH^M, the Austro-Hungarian Table, Males (calculated by Gruder).

In criticizing these figures, allowance has to be made for the fact that the mortality experience upon which the AH^M is based dates from the period 1875–1900.

Austrian Annuity Values—Population and Assured Lives. Interest 3 *per-cent.*

Age	30	40	50	60	70
Population Table .	20·495	17·570	14·207	10·612	7·192
AH^M	20·782	17·605	14·084	10·504	7·217

Graduation.

In the preparation of the Austrian Population Life Tables as finally adjusted, five separate processes are to be distinguished. The following description applies to the experience of males in Austria based on the Censuses of 1900 and 1910, and on the deaths during the years 1906 to 1911.

In order to form a mortality table from population statistics for a series of years, there is required on the one hand the population at the end of each of the intervening years and the deaths during each calendar year. The deaths for each calendar year and the population at the end of 1910 are given, but all other figures are "constructed", and their derivation constitutes (in the wider sense) the first problem in graduation.

The living and the deaths in all observations of population statistics show a tendency to be grouped about the quinquennial ages, and the second operation consists in smoothing away this peculiarity.

An analogous task arises in dealing with the observations at advanced ages. The aggregates of living and deaths aged 99 and upwards are in Austria only recorded in one total. It is necessary to distribute these figures. The irregularities remaining after these operations have been performed are ascribed to errors of observation, which are fairly numerous in population statistics, as for example by incorrect statements of age, unchecked migrations, &c. The elimination of these defects is the duty of graduation in its narrower sense, and this consists, in the present instance, in

(*a*) a preliminary graduation according to some formula for the law of mortality, and

(*b*) a final adjustment or graduation to restore systematic deviations from the results of the observations which may have been caused by the preliminary graduation.

In the preliminary graduation the requirements of practice were kept in mind, and accordingly the Gompertz-Makeham formula was employed.

Thus the procedure of graduation was comprised in the following operations:

1. The completion and smoothing of the observations, and redistribution of the totals at the highest ages (preliminary procedure).

2. The graduation by the Gompertz-Makeham formula (preliminary graduation).

3. The final graduation or adjustment, restoring systematic deviations removed by the preliminary graduation.

In order to characterize more clearly the results of the separate operations, the original mortality table after completing the observations will be described as the "rough" table, which, after smoothing, will become the "smoothed" table. After graduation by the Gompertz-Makeham formula we shall have the "insurance" table, which when finally adjusted will become the "final" table. The "rough" and the "smoothed" tables count as ungraduated, and the "insurance" and "final" tables as graduated.

The Completion and Smoothing of the Observations and Redistribution of the Facts at the highest ages.

If there had been no migrations or losses among the living, it should have been possible to reconcile the living at the beginning of the period with those at the end, with the aid of the numbers of deaths. These relations did not exist exactly. The deviation or balance not accounted for, namely,

$$l^0_x - l^{10}_{x+10} - (m^1_x + m^2_{x+1} + \ldots + m^{10}_{x+9})$$

is termed the migration $J_{x.10}$, where

$l^0_x =$ the population aged x alive at the first Census (1900)

$l^{10}_x =$ the population aged x alive at the last Census (1910)

$m^1_x =$ the deaths in the $(x+1)$th year of age during the first year under observation, &c.

m being used for the deaths among the population to differentiate them from those of a stationary population.

The first process of graduation consists in allocating these variations over the 10 calendar years. This was done by a uniform distribution, thus:

$$l^1_{x+1} = l^0_x - m^1_x + \frac{J_{x.10}}{10},$$

$$l^2_{x+2} = l^1_{x+1} - m^2_{x+1} + \frac{J_{x.10}}{10} \ldots$$

This method was slightly altered for the first ten years of life. For such, by an analogous process, it was calculated how many, out of those born during the period of observation, would still be alive at the end of that period: the decrements, of course,

cannot be divided by 10, but by the number of years from birth until the later census.

It may be remarked that the German investigation 1890–1900 followed the above process, with two modifications:

(1) the graduated value of the migration factor was taken as the arithmetic mean of five neighbouring values;

(2) the allocating of these values to the separate calendar years was not by a uniform distribution but in proportion to the overseas emigrations during the time in question.

The simpler process followed by the Austrians was found however to be adequate. The decrements applicable to the ages under 10 were found to be very fluctuating, and in particular the decrements for 1909 and 1910 could not be explained by emigration alone. It was clear that many children fail to be included in the census returns, a phenomenon brought out in earlier investigations. Nevertheless, although German investigations had made allowance for this, ages 1 and 2 were dealt with in the same way as other ages.

The observations having been completed, the rough table was thus obtained (see table, p. 234).

The Elimination of Quinquennial Humps.

Clearly marked humps are shown in the rough table in heavy type, while possible though less definitely marked humps are indicated by question marks. They are particularly in evidence after age 70, and are exhibited both in the absolute numbers of living and deaths and in the relative numbers—the rates of mortality. It is indispensable to eliminate them before proceeding further, to avoid the risk of bringing them into the final table as typical (as indeed they are, though definitely known as errors).

The method of elimination adopted was to assume that the figure recorded at the hump had gained at the expense of the two neighbouring values, so that, if l_0 be the value recorded at the hump and l_{-1}, l_{+1} be the adjacent values, and λ_{-1}, λ_0, λ_{+1} be the improved values, we have $l_{-1}+l_0+l_{+1}=\lambda_{-1}+\lambda_0+\lambda_{+1}$. It was further assumed that the extent of the disturbance was proportional to the number of the observations, so that $\lambda_{-1}-l_{-1}=cl_{-1}$, and $\lambda_1-l_1=cl_1$: and finally that the middle term was equal to the arithmetic mean of the three observed values:

$$\lambda_0=(l_1+l_{-1}+l_0)\div 3$$

From these four equations are derived the relationships :

$$\lambda_1 = l_1 \,.\, M_3 \div M_2$$

$$\lambda_{-1} = l_{-1} \,.\, M_3 \div M_2$$

$$\lambda_0 = M_3$$

when by M_3 is understood the arithmetical mean of the three, and

$$M_2 = \frac{l_1 + l_{-1}}{2}$$

the arithmetical mean of the two neighbouring values.

The smoothed table obtained by the application of this process is shown in table (p. 235).

The following brief remarks are called for anent this table :

1. It cannot be compared directly as regards living and deaths with the rough table, because it has been based on a radix of a million lives at age 0 as against the two and a half million of the rough table.
2. The comparatively large radix has been used in order to exhibit isolated fluctuations more clearly and to facilitate the further graduation.

The Rates of Mortality at advanced ages.

An extremely difficult problem is presented in ascertaining the rates of mortality for ages over 90. There are few facts, and those unreliable, while for ages 99 and upwards all are grouped together ; and finally there is no check over the deaths by the facts regarding the living. In explanation of the latter point it may be mentioned that the sum of the deaths between two censuses should equal the difference between the living at each census : those living aged between 90 and 100 at the census of 1900 had by 1910 quite died out, or at least, were no longer shown in the Austrian statistics. In the present instance the method selected for dealing with this feature was to extend the Gompertz-Makeham curve, based on the facts for ages 28–76, to the limit of life—this step falling consequently within the scope of the preliminary graduation.

In order, however, that the rates so obtained should fit the observations, the final graduation dealt with the differences between the observed values and those of the preliminary graduation. In constructing the former values, only the living and the deaths were brought in which applied to those calendar

years for which a check is given by the censuses: that is the generations which at the 1900 census appertained to the age groups 80–90 and the deaths up to the census 1910 within the years 1900–1910.

The death totals from the generations 1800–1810 which cannot be checked, were neglected as unreliable observations: thus a reduced material only was utilized.

The difference in the results due to these limited observations is shown by the following summary:

Mortality of the Age-Group 95–100.

Age	Based upon all Deaths						Based upon figures that can be checked					
	Observed		Calculated		Rate of Mortality	Expectation of Life	Observed		Calculated		Rate of Mortality	Expectation of Life
	Living	Deaths	Living	Deaths			Living	Deaths	Living	Deaths		
94	1367	357	1451	379	0·261	...	1367	357	1451	379	0·261	...
95	1090	266	1072	261	0·244	2·54	810	209	1072	277	0·258	2·41
96	865	179	811	168	0·207	2·20	448	103	795	183	0·247	2·08
97	643	114	643	114	0·177	1·65	242	47	612	118	0·194	1·55
98	498	303	529	322	0·608	0·89	113	79	494	346	0·700	0·80
99	...	...	207	207	1·000	0·50	...	...	148	148	1·000	0·50

The rates of mortality from the observations that can be checked are throughout higher, and the expectations of life smaller than those of the full aggregates.

No completely satisfactory explanation is offered for this, but probably it is due to the exaggeration of ages by extremely old people living, which is to some extent corrected in the death records.

It is now possible to pass to the graduation in the narrower sense, first of all by the application of the Gompertz-Makeham formula.

Graduations by the Gompertz-Makeham Formula.

The method of King and Hardy was applied to the observations for ages 28–76 inclusive to obtain $\log c$. The values of the other constants were obtained by the method of Gauss. This consists in making the sum of the squares of the differences between the actual and assumed values of $\log p_x$ a minimum.

As regards the calculations, it is to be observed that in the Gauss formulas likewise only the observations for ages 28–76 were employed. By making use of the values of s, g and c the first graduation for practical insurance purposes was obtained. The radix of the living was so arranged that the commencing number agreed with that of the smooth table at age 27. As a matter of fact this method was not followed precisely. Only c was determined by the King-Hardy method, and A and B by Gauss. This greatly facilitated the calculations. The graduated table follows (p. 236).

The Final Graduation. (*Preliminary Remarks.*)

The above Gompertz-Makeham graduation is satisfactory as regards the continuity of the series and likewise is suitable for practical purposes to an extraordinary degree, between ages 15 and 84. Within these limits the rates of mortality and the expectations of life nowhere deviate seriously from those of the smoothed tables. Lastly, even the principal characteristics of the deaths (the minimum at age 12, the maximum at age 71) are retained.

Nevertheless it fails for more delicate investigations, because it fails to reproduce characteristic features in the rates of mortality or the secondary changes in the deaths series. This exacting condition is said to have been formulated first by Altenburger, in the following terms: "This does not prove "that it is justifiable by any smoothing simply to eliminate "turning points. On the contrary it is of the highest scientific "importance clearly to recognize the waved form of the "mortality curve determined by observation, so that it may be "possible, by repeated comparisons with corresponding material, "to find those characteristics of the mortality curve which "ought not to be smoothed away."

The inadequacy of the Gompertz-Makeham Table for such researches is made more evident by comparing the column of the living with that of the smoothed table. The deviations over long sections have the same sign and hence constitute definite humps (p. 237).

These humps consequently must be regarded as systematic variations, to be attached to the Gompertz-Makeham figures for more advanced scientific purposes. Two processes for so doing were adopted for the present table:

1. Graphic graduation.
2. Graduation by the Bruns series.

Graphic Graduation.

This was carried out by two independent graduators upon squared millimetre paper.

Their results were adopted as the final table where both agreed (as for ages 7, 10, 13, 14, 15, 17, 19–23, 26, 27, 36, 63, 69). In other cases a middle value was chosen, such that in the series of the deaths and the death rate obtained from the column of the living no discontinuities or isolated fluctuations should appear.

The table (p. 240) shows quinquennial values of the graphic graduation finally adopted.

The table agrees practically in all respects with the smoothed table, and the expectations of life with a few exceptions agree as far as the second place of decimals. Moreover, the requisite continuity and agreement at characteristic ages is preserved, such as the minimum death rate at age 14 and the turns at ages 26–29 and 93–96; the minimum of the deaths at age 14 and the maximum at age 71. All the same, the graduation was applied, not to the original observations, but only to the differences between them and the graduated table, and thus permitted scope for judgment by the two graduators, which showed itself at several points (ages 43–53, 74–86).

A second graduation of the humps was made, the series of Bruns being employed for the purpose.

The Graduation by the Bruns Series. The Principles of the Method.

Graduation by the series of Bruns depends upon very simple mathematical assumptions. It is known that generally any statistical function can be represented by so-called infinite mathematical series, that is by series containing an infinite number of terms, such as those of Fourier. A necessary condition is that the remaining terms, at least from one fixed term, should exercise an ever-lessening effect on the function, and that the rest-terms should effect no further alteration.

If now of this infinite series a finite number of terms be employed to represent the function, then the latter will be inaccurate, and, if only a few terms be used, it will be reproduced only as regards its principal features. The fluctuations or irregularities will in general disappear.

By a graduated series is understood a series of observations from which the fluctuations have been eliminated. The attempt can hence be made to graduate a given series of values by

developing the infinite series which reproduces it, only up to a given term.

Such series are the "humps" obtained by graduating a life table by the Gompertz-Makeham formula and subtracting the graduated from the ungraduated numbers of the living.

The Gauss error function with its derivatives fulfils, as does the series of Fourier, the above conditions, for the reproduction of a series (and hence of the "humps"). With reference to other characteristics, the following may be observed:

The results of an experimental series regarding a purely accidental phenomenon which in every individual case strives towards the same final result ξ (the hump maximum) may be arranged according to the Gauss error curve:

$$\psi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2(x-\xi)^2}$$

The latter depends upon only one constant h (the measure of precision) and is valid for experimental attempts on the same object (such as measurement of the same thing objectively) as well as on changing objects, subjectively, such as a mortality table. In a mortality table any one of the humps to be graduated can be regarded as a deviation from the Gompertz-Makeham formula, arising from one or more causes not allowed for in that formula.

For the development of the formulas actually employed, a reference must be made to the original work, where they are set out in full.

The column of the living having been calculated, the final table has yet to be finished, first by completing the living for ages 0 and 1, and then by correcting the final integers of the column to avoid irregularities at the junctions of the humps.

For ages 0 and 1 the values of the ungraduated table were used, and further, the living for ages 2 and 3 were taken from this table, to avoid jumps in the column of deaths at these points.

The Results of the Graduation by Bruns Series.

To answer the question as to how far the graduation has met the requirements of the problem, the first and second differences of the deaths, as well as the differences with the ungraduated column of the living were formed (pp. 239–240).

The graduation fails to be completely satisfactory only in

so far that the minimum of the deaths is transferred to age 12 from age 14, and its value reduced to 162 per 100,000 living at birth instead of 196. This hump was regraduated by a Bruns series with four remainder terms. The resulting curve showed a satisfactory agreement as regards the position and size of the minimum, but the other discontinuities of the curve became more evident. Thus it appears that the Bruns series with two remainder terms represents the maximum efficiency in graduation. Otherwise, the maximum of the deaths agrees with the observations in being placed at age 71, and the number of deaths (1,717 against 1,696) is sufficiently close.

The first and second differences of the deaths run extremely regularly; and nowhere are there any considerable variations between the graduated and ungraduated values. The sum of the positive and negative deviations for a million originally living amounted altogether to 11,588 and 15,174 respectively, the resulting deviations being 3,586.

The turning points appear clearly as secondary minima. Beyond the primary minimum at age 12, the rates of mortality show a secondary minimum at ages 25, 26 and a further one at age 94.

Analogous secondary minima are exhibited in the deaths column.

Conclusion.

To give a retrospect of the work, it has been shown that the original figures of the Austrian Censuses of 1900 and 1910 were combined with the records of births and deaths during the years 1900 to 1911 inclusive to produce observations in a form suitable for the compilation of a set of life tables, one for each sex and for each of the ten calendar years 1901 to 1910 inclusive. Such an extensive set however was not completed, the life tables actually prepared being those covering the quinquennial periods 1901–1905 and 1906–1910 respectively for each sex, and to some extent, for the decennial period 1901–1910.

Special attention is called to the treatment of migrations, by which it was possible to present the observations in such detail.

These tables, like all prepared from such material, contain known inaccuracies, apart from those unknown, and in particular

(*a*) a tendency for observations to be grouped unduly about the quinquennial ages;

(*b*) demonstrably wrong statements as regards very advanced ages.

Accordingly further processes were applied to these "rough" tables, and "smoothed" tables were deduced. The excess observations at the quinquennial ages were distributed over the two neighbouring ages, and the observations for extremely old ages were practically ignored.

The smoothed tables were then graduated by the formula of Gompertz-Makeham, with the object of producing results of practical use in life assurance work, especially for industrial life assurance, collecting societies and pension funds. The value of log c having been obtained by the King-Hardy aggregate methods, based on the observations for ages 28–76, improved values of the other constants were derived by the method of Gauss, which consists in making the sum of the squares of the differences between the actual and assumed values of log p_x a minimum.

These graduations yielded results quite satisfactory for the practical purposes in view, and afford the known advantages of the Gompertz-Makeham formula for actuarial calculations.

As however not only practical insurance purposes but also scientific investigations were expected to be directed to these life tables, further extensive graduations or adjustments were made, in order that the "fit" between the smoothed ungraduated tables and the final results should be as close as possible.

To this end, the differences between the "smoothed" table results and those of the Gompertz-Makeham graduation were subjected to two different kinds of process for the experience 1906–1910:

(*a*) a graphic graduation;

(*b*) a graduation by the Bruns series.

It will be noted that these processes were applied to the differences, and not to the actual life table. As regards the graphic graduation this was done independently by two computers, and an intermediate value selected where variations in their results were found.

The differences between the smoothed and the Gompertz-Makeham tables respectively form a series of "humps", with numbers rising to a maximum and then diminishing. The Bruns series method consists in graduating each of these humps by using a limited number of terms of the well-known Gauss error curve

$$\psi(x)=\frac{h}{\sqrt{\pi}}e^{-h^2(x-\xi)^2}$$

where ξ is the maximum point and h the measure of precision.

A summary of the rates of mortality for this experience is subjoined (p. 241).

The numerical work was very laborious, but the final adjustment produced a close fit.

The processes adopted have been fully described by Professor Dr. Ernst Blaschke, Court Councillor, a corresponding Member of the Institute of Actuaries, in the official Austrian publication entitled "Ergebnisse der Volkszählung vom 31 Dezember 1910 in Österreich", published at Vienna by Karl Gerold's Sohn, 1917.

Professor Blaschke has been good enough to take great interest in these notes and I must express my indebtedness to him for so doing. He has also presented a copy of the volume to the Institute of Actuaries. Furthermore I have to thank my colleague, Mr. Claud W. Wentworth, for his kind assistance in connection with this paper.

Observations, Austrian Male Population, 1906–1911 (*Rough Table*).

Age	Living	Deaths	$100q_x$	Age	Living	Deaths	$100q_x$
0	2,403,462	528,198	21·974	50	656,117	**12,774**	**1·947**
1	1,846,346	97,943	5·304	51	622,732	11,670	1·874
2	1,773,654	42,990	2·424	52	597,728	12,612	2·110
3	1,755,071	26,720	1·540	53	570,906	12,280	2·151
4	1,724,003	18,892	1·096	54	549,179	12,635	2·301
5	1,690,782	14,082	0·833	55	531,482?	13,038	2·453
6	1,679,562	11,391	0·679	56	531,363	13,819	2·601
7	1,648,889	9,544	0·579	57	514,096	13,949	2·713
8	1,620,843	7,842	0·486	58	497,242	14,491	2·914
9	1,578,379	6,492	0·412	59	469,161	14,486	3·088
10	1,561,381	5,778	0·370	60	446,039	**16,615**	**3·725**
11	1,533,916	5,090	0·332	61	420,461	15,019	3·564
12	1,499,973	4,543	0·303	62	408,377	16,078	3·937
13	1,472,661	4,386	0·298	63	385,321	16,334	4·238
14	1,430,872	4,233	0·296	64	376,789	17,455	4·630
15	1,380,499	4,754	0·344	65	354,335	**18,234**	5·146?
16	1,323,753	5,656	0·427	66	344,390	17,887	5·194
17	1,285,216	6,417	0·499	67	316,229	18,021	5·698
18	1,249,542	7,405	0·593	68	296,696	17,941	6·048
19	1,231,116	7,647	0·623	69	272,359	17,848	6·553
20	1,208,877	**8,132**	0·672?	70	252,483	**20,123**	**7·972**
21	1,191,708	8,198	0·688	71	222,290	17,403	7·749
22	1,165,704	8,131	0·698	72	203,458	17,677	8·688
23	1,129,472	7,761	0·687	73	185,165	17,008	9·186
24	1,100,094	7,739	0·708	74	167,195	16,829	10·006
25	1,059,181	7,340	0·687	75	146,250	16,781	**11·474**
26	1,037,100	7,299	0·681	76	135,643	15,425	11·370
27	1,015,914	7,077	0·697	77	116,324	14,232	12·239
28	1,002,961	6,887	0·687	78	99,398	13,427	13·513
29	981,416	6,761	0·689	79	82,965	11,925	14·377
30	974,312	**7,378**	**0·757**	80	70,022	**12,112**	**17·302**
31	963,573	6,878	0·714	81	53,317	9,107	17·081
32	952,955	7,231	0·759	82	45,467	8,453	18·605
33	933,647	7,377	0·790	83	37,301	7,331	19·662
34	917,723	7,575	0·825	84	31,218	6,844	21·993
35	889,871	7,714	0·867	85	23,888	5,713	**23·916**
36	875,475	7,861	0·898	86	20,077	5,618	23·001
37	848,586	7,919	0·933	87	15,101	3,702	24·536
38	832,480	8,207	0·986	88	11,462	3,014	26·381
39	808,454	8,026	0·993	89	8,110	2,196	27·078
40	798,845	**9,144**	**1·140**	90	5,703	1,800	**31·655**
41	777,239	8,625	1·110	91	3,528	1,042	29·535
42	770,042	9,432	1·225	92	2,553	784	30·703
43	753,545	9,129	1·211	93	1,812	544	30·044
44	747,738	9,651	1·291	94	1,365	363	26·593
45	722,377	10,014?	1·386?	95	818	207	25·306
46	719,249	10,081	1·402	96	440	106	23·540
47	700,575	10,366	1·480	97	242	47	19·421
48	689,508	10,916	1·583	98	113	79	69·912
49	673,695	11,959	1·701	99	...	...	...

Mortality Table, Austrian Male Population, 1906–1910.
Ungraduated (Smoothed) Table.

Age	Living	Deaths	$100q_x$	Expectation of Life	Age	Living	Deaths	$100q_x$	Expectation of Life
0	1,000,000	219,820	21·982	40·690	50	448,443	8,958	1·834	19·040
1	780,180	41,365	5·302	51·014	51	479,485	9,245	1·928	18·386
2	738,815	17,909	2·424	52·842	52	470,240	9,922	2·110	17·538
3	720,906	11,102	1·540	53·143	53	460,318	9,901	2·151	17·109
4	709,804	7,595	1·070	52·966	54	450,417	10,310	2·289	16·474
5	702,209	6,116	0·871	52·533	55	440,107	10,765	2·446	15·849
6	696,093	4,629	0·665	51·991	56	429,342	11,210	2·611	15·234
7	691,464	4,004	0·579	51·335	57	418,132	11,344	2·713	14·629
8	687,460	3,341	0·486	50·631	58	406,788	11,854	2·914	14·023
9	684,119	2,812	0·411	49·876	59	394,934	12,705	3·217	13·428
10	681,307	2,527	0·371	49·080	60	382,229	13,264	3·470	12·858
11	678,780	2,247	0·331	48·261	61	368,965	13,662	3·703	12·302
12	676,533	2,050	0·303	47·419	62	355,303	13,989	3·937	11·756
13	674,483	2,010	0·298	46·562	63	341,314	14,465	4·238	11·218
14	672,473	1,964	0·292	45·700	64	326,849	15,303	4·682	10·692
15	670,509	2,380	0·355	44·832	65	311,546	15,477	4·968	10·192
16	668,129	2,806	0·420	43·990	66	296,069	15,760	5·326	9·699
17	665,323	3,320	0·499	43·174	67	280,309	15,980	5·698	9·218
18	662,003	3,926	0·593	42·385	68	264,329	15,987	6·048	8·743
19	658,077	4,126	0·627	41·637	69	248,342	17,088	6·881	8·274
20	653,951	4,316	0·660	40·897	70	231,254	17,162	7·421	7·848
21	649,635	4,515	0·695	40·165	71	214,092	17,273	8·068	7·638
22	645,120	4,503	0·698	39·443	72	196,819	17,099	8·688	7·046
23	640,617	4,401	0·687	38·717	73	179,720	16,509	9·186	6·669
24	636,216	4,440	0·698	37·951	74	163,211	16·641	10·196	6·293
25	631,776	4,436	0·702	37·244	75	146,023	16,023	10·932	5·951
26	627,340	4,391	0·700	36·504	76	130,547	15,371	11·774	5·620
27	622,949	4,342	0·697	35·758	77	115,176	14,096	12·239	5·303
28	618,607	4,250	0·687	35·006	78	101,080	13,659	13·513	4·973
29	614,357	4,349	0·708	34·244	79	87,421	13,188	15·085	4·672
30	610,008	4,417	0·724	33·485	80	74,233	12,118	16·325	4·413
31	605,591	4,408	0·728	32·725	81	62,115	10,813	17·407	4·177
32	601,183	4,557	0·759	31·962	82	51,302	9,544	18·605	3·952
33	596,626	4,720	0·790	31·202	83	41,758	8,211	19·662	3·741
34	591,906	4,889	0·826	30·447	84	33,547	7,351	21·912	3·534
35	587,017	5,066	0·863	29·696	85	26,196	6,051	23·101	3·385
36	581,951	5,249	0·902	28·950	86	20,145	4,821	23·932	5·252
37	576,702	5,381	0·933	28·209	87	15,324	3,760	24·536	3·117
38	571,321	5,633	0·986	27·470	88	11,564	3,051	26·381	2·968
39	565,688	5,798	1·025	26·739	89	8,513	2,391	28·084	2·853
40	559,890	6,058	1·082	26·011	90	6,122	1,856	30·316	2·771
41	553,832	6,308	1·139	25·290	91	4,266	1,272	29·826	2·760
42	547,524	6,708	1·225	24·575	92	2,994	919	30·703	2·720
43	540,816	6,549	1·211	23·874	93	2,075	624	30·044	2·704
44	534,267	6,929	1·297	23·160	94	1,451	379	26·116	2·650
45	527,338	7,177	1·361	22·458	95	1,072	277	25·834	2·411
46	520,161	7,392	1·421	21·761	96	795	183	22·991	2·176
47	512,769	7,589	1·480	21·068	97	612	118	19·421	1·548
48	505,180	7,997	1·583	20·377	98	494	346	70·000	0·801
49	497,183	8,740	1·758	19·696	99	148	148	...	0·500

Mortality Table, Austrian Male Population, 1906–1910, *Graduated. Gompertz-Makeham Formula.*

Age	Living	Deaths	$100q_x$	Expectation of Life	Age	Living	Deaths	$100q_x$	Expectation of Life
0	734,323	4,152	0·565	55·436	50	490,866	8,477	1·728	19·170
1	730,171	4,138	0·567	54·748	51	482,089	8,853	1·836	18·498
2	726,033	4,125	0·568	54·058	52	473,236	9,250	1·955	17·835
3	721,908	4,113	0·570	53·364	53	463,986	9,667	2·084	17·180
4	717,795	4,102	0·571	52·667	54	454,319	10,105	2·224	16·535
5	713,693	4,091	0·573	51·966	55	444,214	10,563	2·378	15·900
6	709,602	4,083	0·575	51·263	56	433,651	11,037	2·545	15·225
7	705,519	4,074	0·577	50·557	57	422,614	11,529	2·728	14·661
8	701,445	4,069	0·580	49·848	58	411,085	12,033	2·927	14·058
9	697,376	4,063	0·583	49·136	59	399,052	12,547	3·144	13·467
10	693,313	4,059	0·586	48·421	60	386,505	13,068	3·381	12·888
11	689,254	4,058	0·589	47·703	61	373,437	13,590	3·639	12·321
12	685,196	4,057	0·592	46·982	62	389,847	14,107	3·920	11·768
13	681,139	4,059	0·596	46·259	63	345,740	14,612	4·226	11·227
14	677,080	4,064	0·600	45·534	64	331,128	15,100	4·560	10·701
15	673,016	4,069	0·605	44·805	65	316,028	15,558	5·923	10·188
16	668,947	4,077	0·610	44·075	66	300,470	15,979	5·318	9·690
17	664,870	4,088	0·615	43·342	67	284,491	16,352	5·748	9·206
18	660,782	4,102	0·621	42·607	68	268,139	16,666	6·215	8·737
19	656,680	4,119	0·627	41·870	69	251,473	16,907	6·723	8·283
20	652,561	4,139	0·634	41·131	70	234,566	17,066	7·275	7·544
21	648,422	4,163	0·642	40·391	71	217,500	17,127	7·875	7·220
22	644,259	4,190	0·650	39·049	72	200,373	17,083	8,525	7·011
23	640,069	4,221	0·659	38·905	73	183,290	16,918	9·231	6·618
24	635,848	4,257	0·670	38·160	74	166,372	16,630	9·995	6·240
25	631,591	4,298	0·680	37·414	75	149,742	16,207	10·823	5·878
26	627,293	4,344	0·692	36·666	76	133,535	15,649	11·719	5·530
27	622,949	4,395	0·705	35·919	77	117,886	14,958	12·088	5·198
28	618,554	4,452	0·720	35·170	78	102·928	14,136	13·735	4·881
29	614,102	4,516	0·735	34·422	79	88,792	13,198	14·863	4·578
30	609,586	4,586	0·752	33·673	80	75,594	12,156	16·080	4·291
31	605,000	4,665	0·771	32·924	81	63,438	11,031	17·390	4·017
32	600,335	4,750	0·791	32·176	82	52,407	9,851	18·797	3·757
33	595,585	4,845	0·814	31·429	83	42,556	8,642	20·307	3·511
34	590,740	4,949	0·834	30·683	84	33,914	7,436	21·926	3·278
35	585,791	5,063	0·864	29·538	85	26,478	6,264	23·656	3·058
36	580,728	5,187	0·893	29·194	86	20,214	5,155	25·503	2·851
37	575,541	5,323	0·925	28·453	87	15,039	4,137	27·470	2·656
38	570,218	5,470	0·959	27·214	88	10,922	3,228	29·560	2·472
39	564,748	5,631	0·997	26·977	89	7,694	2,445	31·775	2·300
40	559,117	5,805	1·038	26·244	90	5,249	1,791	34·116	2·138
41	553,312	5,994	1·083	25·514	91	3,458	1,265	36·581	1·986
42	547,318	6,197	1·132	24·788	92	2,193	859	39·170	1·844
43	541,121	6,418	1·186	24·066	93	1,334	557	41·878	1·709
44	534,703	6,656	1·245	23·349	94	775	346	44·698	1·580
45	528,048	6,910	1·309	22·637	95	429	204	47·624	1·454
46	521,138	7,183	1·378	21·931	96	225	114	50·645	1·321
47	513,955	7,477	1·455	21·230	97	111	60	53·746	1·160
48	506,478	7,789	1·538	20·536	98	51	29	56·912	0·924
49	498,689	8,123	1·629	19·849	99	22	22	...	0·500

Age	The numbers of the Ungraduated Living compared with the Graduated are		Age	The numbers of the Ungraduated Living compared with the Graduated are	
	Greater by	Smaller by		Greater by	Smaller by
0	265,677	...	51	...	2,604
1	50,009	...	52	...	2,996
2	12,782	...	53	...	3,668
3	...	1,002	54	...	3,902
4	...	4,991	55	...	4,107
5	...	11,484	56	...	4,309
6	...	13,509	57	...	4,482
7	...	13,055	58	...	4,297
8	...	13,985	59	...	4,118
9	...	13,257	60	...	4,276
10	...	12,006	61	...	4,472
11	...	10,474	62	...	4,544
12	...	8,663	63	...	4,426
13	...	6,656	64	...	4,279
14	...	4,607	65	...	4,482
15	...	2,507	66	...	4,401
16	...	818	67	...	4,191
17	453	...	68	...	3,810
18	1,221	...	69	...	3,131
19	1,397	...	70	...	3,312
20	1,390	...	71	...	3,408
21	1,213	...	72	...	3,554
22	861	...	73	...	3,570
23	548	...	74	...	3,161
24	368	...	75	...	3,172
25	185	...	76	...	2,988
26	47	...	77	...	2,710
27	...	...	78	...	1,848
28	53	...	79	...	1,370
29	255	...	80	...	1,361
30	422	...	81	...	1,323
31	591	...	82	...	1,105
32	848	...	83	...	798
33	1,035	...	84	...	367
34	1,166	...	85	...	282
35	1,226	...	86	...	69
36	1,223	...	87	265	...
37	1,161	...	88	642	...
38	1,103	...	89	819	...
39	940	...	90	873	...
40	773	...	91	808	...
41	520	...	92	801	...
42	206	...	93	741	...
43	...	205	94	676	...
44	...	437	95	643	...
45	...	710	96	570	...
46	...	977	97	501	...
47	...	1,186	98	443	...
48	...	1,298	99	126	...
49	...	1,506	100	...	...
50	...	2,123	...	...	...

Graduated Mortality Table.

Age	Living	Deaths	$100q_x$	Expectation of Life	Age	Living	Deaths	$100q_x$	Expectation of Life
0	100,000	21,982	21·982	40·693	55	44,062	1,082	2·456	15·828
1	78,018	4,137	5·303	51·018	56	42,980	1,127	2·622	15·214
2	73,881	1,791	2·424	52·847	57	41,853	1,171	2·798	14·610
3	72,090	906	1·257	52·147	58	40,682	1,219	2·996	14·016
4	71,184	743	1·044	52·817	59	39,463	1,267	3·211	13·434
5	70,441	712	1·011	52·369	60	38,196	1,315	3·443	12·863
6	69,729	606	0·869	51·899	61	36,881	1,363	3·696	12·304
7	69,123	468	0·677	51·349	62	35,518	1,412	3·975	11·757
8	68,655	346	0·504	50·696	63	34,106	1,459	4·278	11·223
9	68,309	262	0·384	49·950	64	32,647	1,504	4·607	10·702
10	68,047	207	0·304	49·141	65	31,143	1,548	4·971	10·195
11	67,840	174	0·256	48·289	66	29,595	1,586	5·359	9·702
12	67,666	162	0·239	47·412	67	28,009	1,622	5·791	9·223
13	67,504	184	0·273	46·525	68	26,387	1,650	6·253	8·759
14	67,320	232	0·345	45·650	69	24,737	1,671	6·755	8·310
15	67,088	279	0·416	44·807	70	23,066	1,685	7·305	7·875
16	66,809	310	0·464	43·992	71	21,381	1,689	7·900	7·457
17	66,499	333	0·501	43·194	72	19,692	1,682	8·542	7·053
18	66,166	362	0·547	42·409	73	18,010	1,665	9·245	6·665
19	65,805	402	0·611	41·640	74	16,345	1,635	10·003	6·294
20	65,402	435	0·665	40·893	75	14,710	1,592	10·823	5·937
21	64,967	451	0·694	40·163	76	13,118	1,538	11·724	5·597
22	64,516	451	0·699	39·440	77	11,580	1,468	12·677	5·274
23	64,065	448	0·699	38·714	78	10,112	1,386	13·706	4·967
24	63,617	443	0·696	37·984	79	8,726	1,295	14·841	4·677
25	63,174	439	0·695	37·246	80	7,431	1,193	16·054	4·405
26	62,735	436	0·695	36·504	81	6,238	1,082	17·345	4·152
27	62,299	434	0·697	35·756	82	5,156	968	18·774	3·918
28	61,865	434	0·702	35·003	83	4,188	849	20·272	3·708
29	61,431	435	0·708	34·247	84	3,339	727	21·773	3·524
30	60,996	437	0·716	33·487	85	2,612	605	23·162	3·365
31	60,559	444	0·733	32·725	86	2,007	492	24·514	3·229
32	60,115	455	0·757	31·963	87	1,515	389	25·677	3·115
33	59,660	469	0·786	31·203	88	1,126	300·4	26·643	3·019
34	59,191	485	0·819	30·447	89	825·5	226·8	27·482	2·933
35	58,706	504	0·859	29·694	90	598·7	169·0	28·214	2·856
36	58,202	524	0·900	28·947	91	429·7	123·4	28·800	2·781
37	57,678	545	0·945	28·205	92	306·3	88·3	28·800	2·706
38	57,133	570	0·998	27·469	93	218·0	62·7	28·900	2·596
39	56,563	595	1·052	26·741	94	155·3	44·1	28·400	2·448
40	55,968	614	1·097	26·020	95	111·2	32·9	29·700	2·221
41	55,354	629	1·136	25·303	96	78·3	26·1	33·300	1·949
42	54,725	645	1·179	24·588	97	52·2	20·6	38·500	1·673
43	54,080	664	1·228	23·876	98	32·0	14·5	43·800	1·306
44	53,416	686	1·284	23·166	99	17·5	9·8	55·600	1·111
45	52,730	712	1·350	22·461	100	7·7	5·7	75·000	0·875
46	52,018	743	1·428	21·762	101	2·0	1·4	...	0·800
47	51,275	774	1·509	21·070	102	0·6	0·6	...	0·500
48	50,501	808	1·600	20·385	...	...	...	...	...
49	49,693	842	1·694	19·708	...	...	...	...	...
50	48,851	880	1·801	19·039	...	...	...	...	...
51	47,971	917	1·912	18·380	...	...	...	...	...
52	47,054	957	2·034	17·728	...	...	...	...	...
53	46,097	997	2·163	17·086	...	...	...	...	...
54	45,100	1,038	2·302	16·452	...	...	...	...	...

Test of the Graduation by the Differences of the Deaths and also of the Living by the Graduated and Ungraduated Tables.

Age	Calculated Deaths	First		Second		Compared with the Graduated Table the Smoothed Table is	
		Differences of the Deaths					
		+	−	+	−	Larger	Smaller
4	7,422	...	...	...	...	...	2,033
5	7,122	...	300	...	...	...	2,205
6	6,064	...	1,058	758	...	...	1,199
7	4,683	...	1,381	323	...	236	...
8	3,459	...	1,224	...	157	915	...
9	2,612	...	847	...	377	1,033	...
10	2,075	...	537	...	310	833	...
11	1,737	...	338	...	199	381	...
12	1,627	...	110	...	228	...	129
13	1,835	208	...	318	...	...	552
14	2,319	484	...	276	...	...	727
15	2,788	469	...	...	15	...	372
16	3,102	314	...	...	155	36	...
17	3,335	233	...	...	81	332	...
18	3,617	282	...	49	...	347	...
19	4,016	399	...	117	...	38	...
20	4,356	340	...	...	9	...	72
21	4,503	147	...	...	593	...	32
22	4,515	12	...	...	135	...	44
23	4,478	...	37	...	149	...	32
24	4,432	...	46	...	9	...	55
25	4,390	...	42	4	...	37	...
26	4,355	...	35	7	...	...	9
27	4,340	...	15	20	...	...	45
28	4,342	2	...	17	...	...	47
29	4,348	6	...	4	...	45	...
30	4,375	27	...	23	...	44	...
31	4,441	66	...	39	...	2	...
32	4,547	106	...	40	...	35	...
33	4,689	142	...	36	...	19	...
34	4,856	167	...	25	...	...	6
35	5,037	181	...	14	...	...	39
36	5,239	202	...	21	...	...	68
37	5,452	213	...	11	...	...	78
38	5,699	247	...	34	...	...	7
39	5,950	251	...	4	...	59	...
40	6,138	188	...	...	63	211	...
41	6,291	153	...	...	35	291	...
42	6,449	158	...	5	...	274	...
43	6,638	189	...	31	...	15	...
44	6,862	224	...	35	...	104	...
45	7,125	263	...	39	...	37	...
46	7,422	297	...	34	...	...	15
47	7,742	320	...	23	...	15	...
48	8,079	337	...	17	...	168	...
49	8,428	349	...	12	...	250	...
50	8,794	366	...	17	...	...	62
51	9,173	379	...	13	...	...	226
52	9,564	391	...	12	...	...	298
53	9,970	406	...	15	...	...	656
54	10,387	417	...	11	...	...	587

Continued next page.

Test of the Graduation by the Differences of the Deaths and also of the Living by the Graduated and Ungraduated Tables—continued.

Age	Calculated Deaths	First — Differences of the Deaths +	First — Differences of the Deaths −	Second — Differences of the Deaths +	Second — Differences of the Deaths −	Compared with the Graduated Table the Smoothed Table is Larger	Compared with the Graduated Table the Smoothed Table is Smaller
55	10,821	434	...	17	...	...	510
56	11,262	441	...	7	...	...	454
57	11,719	457	...	16	...	...	402
58	12,189	470	...	13	...	...	27
59	12,665	476	...	6	...	308	...
60	13,147	482	...	6	...	268	...
61	13,635	488	...	6	...	151	...
62	14,116	481	...	...	5	124	...
63	14,589	472	...	...	8	251	...
64	15,045	456	...	...	17	375	...
65	15,476	431	...	...	25	117	...
66	15,866	390	...	...	41	116	...
67	16,213	347	...	...	43	213	...
68	16,500	287	...	...	60	455	...
69	16,715	215	...	...	72	968	...
70	16,849	134	...	...	81	595	...
71	16,889	40	...	...	94	282	...
72	16,825	...	64	...	104	...	102
73	16,648	...	177	...	113	...	826
74	16,352	...	296	...	121	...	237
75	15,921	...	431	...	135	...	526
76	15,375	...	546	...	115	...	628
77	14,679	...	696	...	150	...	624
78	13,858	...	821	...	125	...	41
79	12,951	...	907	...	56	158	...
80	11,929	...	1,022	...	115	...	79
81	10,828	...	1,101	...	79	...	272
82	9,671	...	1,157	...	56	...	253
83	8,495	...	1,176	...	19	...	126
84	7,266	...	1,229	...	53	158	...
85	6,051	...	1,215	14	...	73	...
86	4,920	...	1,131	84	...	73	...
87	3,893	...	1,027	104	...	172	...
88	3,004	...	889	138	...	305	...
89	2,268	...	736	153	...	258	...
90	1,690	...	578	158	...	135	...
91	1,234	...	456	122	...	...	31
92	883	...	351	105	...	...	64
93	627	...	256	95	...	...	105
94	441	...	186	70	...	...	102
95	329	...	112	74	...	...	40
96	261	...	68	44	...	12	...
97	202	...	59	9	...	90	...
98	145	...	57	2	...	174	...
99	98	...	47	10	...	...	27
100	57	...	41	6	...	...	77
101	14	...	43	...	2	...	20
102	6	...	8	35	...	...	6
...	...	...	6	2	...	...	...
...	...	...	...	6	...	...	...
					Totals	11,588	15,174
						$\underbrace{\hspace{6em}}_{3,586}$	

Austrian Population, Males, 1906–1910.
100 q_x.

Age*	Rough and Smoothed Table	Insurance Table (Gompertz-Makeham Graduation)	Final Graduation: Graphic	Final Graduation: Bruns
2	2·424	0·568	...	...
7	0·579	0·577	0·573	0·677
12	0·303	0·592	0·303	0·239
17	0·499	0·615	0·503	0·501
22	0·698	0·650	0·705	0·699
27	0·697	0·705	0·698	0·697
32	0·759	0·791	0·763	0·757
37	0·933	0·925	0·936	0·945
42	1·225	1·132	1·198	1·179
47	1·480	1·455	1·538	1·509
52	2·110	1·955	2·019	2·034
57	2·713	2·728	2·777	2·798
62	3·937	3·639	3·967	3·975
67	5·698	5·318	5·802	5·791
72	8·688	7·875	8·571	8·542
77	12·239	11·719	12·659	12·677
82	18·605	18·797	18·572	18·774
87	24·536	27·470	25·683	25·677
92	30·703	39·170	28·852	28·758
97	19·421	53·746	32·308	38·462

* These ages taken in order to avoid the exaggerated quinquennial values of the rough table.

Rates of Mortality in Austrian Provinces.

Age	Males						Females					
	Vienna		Sudete		Carpathia		Vienna		Sudete		Carpathia	
	1870/80	1910	1870/80	1910	1870/80	1910	1870/80	1910	1870/80	1910	1870/80	1910
0	23·097	18·500	28·202	20·392	27·431	22·496	20·165	15·598	23·723	17·032	23·732	18·760
1	9·879	4·261	7·131	4·181	8·976	6·884	9·606	4·161	6·822	4·056	8·539	6·727
2	6·413	1·736	3·934	1·604	6·515	3·450	5·930	1·721	3·775	1·511	6·229	3·335
3	4·371	1·019	2·699	0·911	4·753	2·266	4·066	1·072	2·585	0·915	4·453	2·196
4	2·992	0·733	2·075	0·615	3·871	1·630	3·011	0·875	1·983	0·633	3·620	1·564
5–9	1·242	0·379	0·908	0·369	1·862	0·833	1·262	0·358	0·895	0·395	1·783	0·904
10–19	0·870	0·338	0·489	0·356	0·843	0·462	0·873	0·352	0·497	0·401	0·859	0·533
20–29	1·748	0·581	1·070	0·682	1·120	0·682	1·302	0·550	0·828	0·729	1·037	0·750
30–39	2·056	0·968	1·142	0·851	1·416	0·812	1·468	0·704	1·061	0·824	1·596	0·977
40–49	2·698	1·587	1·612	1·354	2·239	1·333	1·666	0·998	1·246	0·985	2·145	1·230
50–59	3·902	2·842	2·583	2·378	4·035	2·487	2·520	1·743	2·117	1·668	3·971	2·265
60–69	5·977	5·475	4·667	4·586	6·738	5·212	4·507	3·909	4·394	3·870	7·304	5·688
70–79	10·750	10·209	8·950	9·987	12·692	10·890	9·300	8·861	9·853	9·063	13·355	11·348
over 80	17·450	20·666	19·911	20·528	20·160	18·121	18·070	18·752	20·125	19·213	20·315	19·266

Abstract of the Discussion.

Mr. V. P. A. DERRICK said that the first thing one looked for in population data was the existence of errors due to mis-statements of age. The surprising thing about the table giving the Austrian data was that those errors appeared in the deaths column only. But that appearance was misleading. The population in the table was not a census population. It was a population made up to meet the deaths of ten years. In the census figures the errors were just as plainly marked as in the deaths.

By the census method used in Austria any difference between the calculated population at the preceding census, as obtained by the addition of deaths to the population recorded at the new census, and the actual population at the preceding census was called "migration." It was quite clear that that difference was not only migration. It included migration, if any, but it also included the errors due to mis-statements of age. When at any age there was an error in one census and no corresponding error at the age ten years earlier in the other, the difference contributed to the migration figure. Similarly with the under-statement of the number of children aged 0. It was clear from the Austrian figures that a considerable number of children were not recorded, or were wrongly recorded, at ages 0 and 1. There were not enough of them as compared with the number of births in the preceding year. Whether they had been recorded later or not, he had not been able to find out; but at any rate there was a deficiency of children enumerated at age 0 and 1. The precise effect of the Austrian method on this class of errors was not quite clear; it depended on the size of the errors. If the errors were increasing according to age the effect of the method was to distribute them at ages rather earlier than the census age, and where they were decreasing the effect was to distribute them at ages rather higher than the enumerated age. So that, as compared with the distribution of the errors in the deaths equally on each side of the age, there was a little difference. It was not very much, but it was one of those small points of difference which seemed to be worth calling attention to. It might be mentioned that the number of children under 2 brought out as migrations by the method was more than double the number of women between the ages of 20 and 45 who migrated during that period. Another point of interest in connection with the rough table was that by the method of collecting the statistics, namely, by years of birth instead of age, and, in the case of deaths, by both year of birth and age, q_x was obtained direct without the intermediate use of m_x.

The graduation was in two stages, the first stage being to obtain a base curve following the Gompertz-Makeham hypothesis, and the second to correct the base curve by a series of other curves super-imposed upon it. If the Gompertz-Makeham graduation had been used solely as a means of getting the final graduation, there would have been nothing more to be said, but it was put forward as something more than this, namely, as an insurance table which, as it was labelled Gompertz-Makeham, they would be willing to

accept. He did not know why Dr. Blaschke should suppose that a table less accurate and less in accordance with facts should be more satisfactory to the actuary—and it was the actuary who had to deal with practical problems—than the final table which he ultimately produced. In the first place it did not fit the facts. If the differences between the graduated table and the ungraduated table for the ages on which the graduation was based, viz., 27 to 76, were examined, it would be observed that the curve, while it coincided with the unadjusted facts at age 28, only once went near the curve afterwards, namely, at age 42. If a Gompertz-Makeham table was desirable, he had no doubt that a better one could have been produced. He thought the reason why the curve was not satisfactory was that the constant c was obtained by the King-Hardy method, log c being ·03723, whereas the other constants were calculated by the method of least squares. It seemed to him that a better way would have been to obtain all the constants from a series of ordinates so that in any case the curve would go near to the original figures at a number of points. However, the curve having been obtained in the manner described it was used for the whole table from age 0. As a base line curve to be corrected afterwards that might be all right; but the Austrian report gave not only the mortality functions, but also commutation columns and annuity values down to age 0, based upon the Gompertz-Makeham assumption, with the result that the expectation at age 0 on the Gompertz-Makeham hypothesis was shown to be 55·4, and on the final table, which was fairly reliable, 40·6. This seemed very ridiculous.

As regards the final process, the resulting table appeared to satisfy the two conditions which were normally expected of such graduations, namely, the tests of smoothness, and of agreement between the expected and actual deaths throughout the table. The changes of sign were frequent, and the total seemed to be in reasonable relationship with the total deaths. The second differences seemed to be perfectly satisfactory. It was mainly, therefore, with regard to the Gompertz-Makeham Table that he thought objection might be raised.

He would like to add one word in regard to the adequacy of the conventional method employed by actuaries in constructing life tables. The actuary was almost solely concerned with the mortality of the future, and he only used the mortality of the past as a guide in making estimates for the future. What was the first lesson to be learned from the examination of the mortality of the past? It was that mortality had been decreasing at almost every age for a great number of years, and not only decreasing, but decreasing at an appreciable rate in many age groups. Their tables were out of date, and they were never in date. They were dead before they were born. He suggested that it was their treatment of them that was wrong. What was really wanted was a table by which the mortality of the past was projected into the future, and which would anticipate, so far as was possible on the evidence of the past, the changes which

might be expected in the future. He raised the point now because of the general overhaul of their statistical machinery which was at present going on. There was an Institute table in preparation; there had been a census last year from which further English mortality tables might be expected, and the Text-Book was being revised and brought up to date, and it did seem to him that before their ideas were stabilized and conventionalized, as they would be for many years, it was desirable to consider whether it was not possible to produce tables which would correspond more nearly to practical requirements.

Mr. F. A. MENZLER thought that large systematic errors in age would probably be more marked in Austria than in this country. There was, too, the supposed preference on the part of females for being recorded in the age group 20-30, as to which the balance of opinion in this country was that the practice still existed. Then there was the case of the older workers, who, for the sake of securing employment, commonly understated their ages, but who would give their correct ages later on if an Old Age Pensions Scheme were in existence. And, finally, although not of the same nature as the foregoing, there were errors in the numbers of young children enumerated. In view of all these considerations, it would seem that the graduated corrections super-imposed on the Makeham curve did not consist merely of true features in the experience but were affected by systematic errors. If adherence to the original data was to be the criterion, he preferred, on the whole, Mr. King's methods of securing it. Whether a mere graduation of ages or a National Life Table were under consideration, some preliminary adjustment was generally considered necessary to deal not only with the preference for digits, but also with the major mis-statements of age, and he would direct attention as indicating a possible line of adjustment to a somewhat neglected investigation by Dr. Dunlop (*J.R.S.S.*, vol. lxxix, p. 309), in which the ages of young children as recorded in the Census returns in a not inconsiderable sample were compared with the ages as deduced from the registers of birth. Dr. Dunlop concluded that the number of children recorded at the Census as under 1 year of age was too small by approximately 5 per-cent, that the number between 1 and 2 was too small by approximately 3 per-cent, and that the number recorded at age 2 last birthday was approximately correct. That result, so far as it went, threw some doubt on the assumptions that had been made in dealing with the early ages. A similar procedure might possibly be adopted for other ages with a view to throwing light on the larger misstatements that were believed to exist. It must be admitted that Dr. Dunlop was not optimistic as to that, but it had occurred to him (the speaker) that the ages on the Census schedules might be compared with those recorded for certain administrative purposes. For example, the Old Age Pension Officer had to obtain evidence of age, and at the last Census over 156,000 returned themselves as old age pensioners. Again, possibly the Ministry of Pensions had data relating to the ages of war widows. While it

might be objected that those were not proper samples, it was nevertheless possible that valuable indications as to the extent of major misstatements of age and also of the preference for digits could be obtained in that way. Then again, the ages of retired officers of the Army and Navy, numbering at the 1911 Census some 10,000. could, he imagined, be compared with the ages appearing in the War office and Admiralty records.

Mr. W. P. ELDERTON agreed with Mr. Derrick's criticism of the method of graduation. If the Makeham graduation were compared with the ungraduated figures, it would be seen that it was impossible to regard the former as a graduation of the facts. It seemed to him that Dr. Blaschke might just as well have taken any other arbitrary series as a base line. He might, for example, have used a zero base line, *i.e.*, have graduated the facts directly, although, of course, the final result could not have been obtained in that case by superimposing five Bruns Charlier curves on the Makeham curve. He much preferred the methods used in this country, especially those of Mr. King, to those indicated in the Paper.

It had been remarked that the final graduation, arrived at by means of the Bruns Charlier curves, agreed very closely with the original facts. If the total number of constants used in that graduation were added up they would be found to be something like 20, and it was not therefore surprising that the graduation should represent the facts closely. He had no sympathy with the desire to make every mortality experience conform to the Makeham type. His other criticisms were both criticisms of Mr. Richmond, although one of them was perhaps a compliment rather than a criticism. They were that the paper was much too long, and that he would have welcomed a short and clear criticism of the method by the author of the Paper. Personally, he preferred English criticism to foreign mathematics.

Mr. A. HENRY thought that an interesting feature of the Austrian method was that it combined the life office method of obtaining the exposed-to-risk and the census method. As Mr. Derrick had pointed out, that had the effect of smoothing away part of the population errors. The method might prove to be a useful tool, but he thought it would be found that it depended on a number of very divergent factors, so that, on the whole, though it was a method which deserved investigation because of its novelty, he doubted whether it would prove to be better than the more straightforward methods to which British actuaries were accustomed.

He was glad to hear Mr. Elderton's remark on the abuse of the Makeham method of graduation. There had been numerous instances—not only in this country but in America, and in the present paper—where, for the inadequate reason of its value for joint life annuity purposes, an attempt had been made to use a Makeham curve when it was quite unsuitable. It led to bad graduation and to wrong conclusions. As to the method employed by Dr. Blaschke he did not understand how the method of least squares (which assumed that the differences in the constants were

merely the result of sampling) could possibly apply to a graduation of the sort under discussion, where clearly the curve did not fit the facts, and where obviously the errors were not the result merely of small numbers. In those circumstances the method of least squares became a purely empirical method of obtaining the constants: it had its advantages but it had no theoretical value. He would prefer on the whole, as an empirical method, the method which Sir George Hardy had accustomed them to, namely, making the first and second sums of the deviations vanish. He did not believe that one could necessarily expect to hit upon the best Makeham graduation at the first attempt. As Hardy had shown in his lectures, one or two trials had to be made with different constants, and it might be found that the least probable value was the one which would give the best practical result.

Mr. F. B. WYATT said that it would have been interesting if Mr. Richmond had given some comparisons of the results of the investigation with the population statistics of other countries. He noticed, for example, a very great difference between the mortality of males and females. In 1910 there was a marked difference after age 30. There was also a great difference in 1910 between the mortality of Vienna and the mortality of Carpathia. Did the statistics apply to the whole of the former Austrian-Hungarian Empire or merely to Austria?

Mr. S. J. H. W. ALLIN said that Mr. Richmond had chosen a very suitable time for his paper, as the results of the recent Census would shortly be available and the task of preparing population tables would have to be undertaken, so that any information with regard to new methods of dealing with population statistics was of interest either as suggesting new processes which might be adopted or as indicating what should be avoided.

He thought that Dr. Blaschke's smoothing method was not satisfactory (1) because migration was assumed to be spread evenly over ten years of age, whereas emigration might have predominated during the early ages and immigration at the older ages, in which case the estimated exposed-to-risk for an intercensal year might be widely inaccurate; (2) because systematic errors of age were included in migrations. With regard to the second of these reasons, it was somewhat unfortunate that the crude figures of the estimated exposed-to-risk were not available, as probably much light would have been thrown on the subject by a study of them. Probably that was more especially the case with regard to women, where the under-statements of age were probably considerably greater than in the case of males.

With regard to the graduation, the author said that the tables were graduated by the Makeham method for the practical purposes of industrial assurance, collecting societies and pension funds. He (the speaker) did not know what the facts were in Austria, but in this country joint life assurances in industrial companies were not very numerous, and there did not seem much object in the use of a Makeham graduation for that purpose. As regards pension

funds, he thought the mortality of a pension fund generally coincided with that of the general population. He doubted whether a Makeham graduation was suitable for population tables at all. With regard to the Bruns graduation, the final graduation, if the actual deaths were compared with the expected deaths the differences were very much smaller than the expected deviations, irrespective of sign. He thought, therefore, that the graduation adhered too closely to the original data, and that systematic errors had not been entirely eliminated.

On the general question of the compilation and graduation of population life tables, he suggested the following points for consideration. In compiling population tables, it was necessary to show what the actual mortality was, and for that purpose he felt that to take the mortality over the whole intercensal period was not of any particular value. It was preferable on the whole, to construct a table from the census data and the deaths of the two or three adjoining years. Such a table gave the latest rates of mortality, and at the same time the problem of migration was considerably reduced. The fact must not, however, be overlooked that the mortality might have been abnormal during the selected years. If, for instance, the recent census had been taken in 1920, and a table had been prepared on the basis of that census and the deaths for 1919–1920, the mortality would have been overestimated owing to the effect of the very heavy influenza epidemic of those two years. Similar influences might affect infantile mortality, which was particularly sensitive to climatic influences. He admitted, however, that this raised a controversial point, since if one began a process of selection one might prejudge the issue. Another point which had to be borne in mind was the question of misstatements of age. That question ought, he thought, to be more fully considered in future than in the past. As regards graduation, a National Mortality Table was one of very great importance and he suggested that it would be worth while to make graduations by two or three different methods, namely, by a mathematical formula, by a summation method, and even by the graphic method. He felt sure that a comparison of the results, by various methods, would throw much light on the subject.

The actual tables of mortality given in the paper were not perhaps of much interest to actuaries. Since the date of the census the Austrian Empire had been dismembered, and moreover the Empire included such a variety of people that one census table did not apply to it as a whole. There was one point, however, to which he would draw attention, namely, the decline shown in the mortality between ages 23 and 26. At first sight one was inclined to think it was due to faulty graduation or to wrong statements of age, but if the English Life Table were referred to it would be seen that at those ages, although there was not an actual reduction in mortality there was a very marked retardation of the increase. Possibly in the past enough attention had not been given to this. In the case of the Austrian tables it might possibly be associated with the heavy rate

of mortality exhibited at the younger ages. It did not appear from the paper what the cause of that heavy mortality was, but it might be typhus or some similar disease which had worn itself out to a certain extent and had left the survivors healthier than those living in the years just previous.

The PRESIDENT, in moving a vote of thanks to Mr. Richmond for his paper, reminded the members that the present was not the first occasion on which they were indebted to Mr. Richmond's intellectual and linguistic gifts for knowledge as to what was happening in other countries than their own. It had been only a few years ago that Mr. Richmond had given them an extremely interesting paper on the methods employed for the construction of certain schemes of national insurance in Austria, and the present table, in effect, completed Mr. Richmond's work so far as the old Austria was concerned; for it was abundantly clear that in regard to any future publications emanating from that unhappy country the whole basis and scope of the material dealt with would be vastly different from what it had been up to the fateful year 1914.

He had been extremely interested to hear from Mr. Derrick's remarks that, despite the apparent regularity in the population column of the ungraduated life table, there was, as indeed he had suspected, precisely the same kind of irregularity at the decennial and other favourite stages as was discernible in the deaths, and, having regard to the methods by which the irregularities had been eliminated, and to the interesting process—quite novel he thought—by which the exposed-to-risk had been constructed, a little more information as to the extent of the so-called "migrations" at various ages would have been useful. The Austrian method could not be used in constructing a national life table for this country, because our census authorities had decided to ask for ages and not for years of birth. The method of the Austrian actuaries and statisticians was based upon the year of birth, and the q_x was found, as Mr. Derrick had pointed out, by tracing the lives through years of age. Although this could not be done with our present data, the Austrian method would add to actuarial information here and might afford a reason for reviewing our own censal methods on future occasions.

Mr. Derrick had made an interesting suggestion as to the kind of life table that would be useful in a period of continually decreasing mortality. That reminded him of a precedent which he thought he had mentioned before. Under the Welsh Disestablishment Act annuity values were to be calculated in respect of the incumbents existing in Wales at a certain date, with a view to the commutation of the values of their incumbencies and the payment of a capital sum to the Church in lieu thereof, on the lines of the Irish Disestablishment Act. But the Welsh Church Bill provided that in ascertaining the value of the life interests of the incumbents regard was to be had, not only to the superior longevity prevailing among the clergy as compared with the general population, but also to the probability of a general decline in the death rate of the community

as a whole. He rather thought that for that far-sighted provision in the interests of the Church in Wales Parliament had been a good deal indebted to the advice of Mr. Wyatt. He believed it was Mr. Wyatt's technical knowledge and his sound judgment on those matters that had suggested to the Government of the day the advisability of making an equitable provision of that kind. The problem of giving effect to the probability of future improvement in the rate of mortality was not by any means an easy one. A double-entry table was necessary, because one had to take account of the rate of mortality at each age at the time when that age would be attained.

He agreed with the views which had been expressed regarding the unsuitability of the Makeham Table for work of the kind under discussion. The Makeham Table was merely a mathematical device by which joint life functions could be readily calculated and a smooth curve obtained. Beyond that the method had no special validity, and when they were told that it was deemed to be the practical method for the purposes of life assurance, industrial life assurance, collecting societies and pension funds, he wondered what there was in the operations of those institutions to vest the Makeham curve with greater authority than any ordinary summation formula which gave a smooth graduation. Knowing how their friends abroad concentrated their attention upon our Text-Book, he thought sometimes that their belief in the Makeham method of graduation might possibly be due to the fact that it occupied a somewhat prominent place in that work. They seemed to regard it as something which represented the views of London as to how a graduation ought to be performed. He hoped that night's discussion would dispel that idea.

He noticed Mr. Richmond did not give any table by which they could test the goodness of the graduation. It seemed to him that the values of e and q at individual ages were not the best instruments by which to measure the effect of graduation. Personally he much preferred such a function as the probability of dying within the next ten years. He had had the curiosity to work out the probability of dying within the next ten years by the Gompertz-Makeham Table and the finally graduated table. At age 10 the Gompertz-Makeham gave a probability of dying within ten years of about 50 per-cent in excess of the probability given by the finally graduated table. From age 35 to age 65 the probability of dying within ten years was less by the Gompertz-Makeham Table than it was by the finally graduated table. The difference might not be large, but the effect was cumulative, and was brought out in the relative values of the expectation of life. For practical purposes the finally graduated table was much better than the Gompertz-Makeham intermediate graduation.

Mr. Richmond, in reply, said that exigencies of space had prevented him from giving a detailed criticism of the various processes employed by Dr. Blaschke. He had thought it best in the circumstances to confine himself to as fair and yet as brief an account as he could prepare of the general lines on which the investigation had been conducted.

The question of projecting mortality into the future had been considered by the German Office, the Gotha, and in a Paper read before the Faculty of Actuaries he (the speaker) had given some account of the methods which had been adopted with the object of estimating future mortality from past experience and particularly with a view to determining whether reserves were likely to be materially affected by a continued change in the rates of mortality. Some interesting investigations into changes in mortality with the lapse of time has also been published by Dr. Pesch. He thought that much of what was called selection was not pure selection, but a mixture of selection with the effect of improvements in sanitary conditions and health. The Swedish writer, Dr. Lundgren, had dealt with that subject some years ago and had shown that the assurance companies, if they valued by select tables, were setting up reserves that were too large.

On the Valuation of Endowment Assurances by Select Tables.

By Edward Harold Brown, F.I.A., *of the Prudential Assurance Company.*

[Submitted to the Institute, 1 May 1922.]

THE question whether it is necessary to use select mortality rates in the calculation of office reserves was pronounced by Mr. R. P. Hardy during the discussion on the papers submitted by Messrs. King, Diver and Ackland to be a highly debatable one.

The discussion disclosed two lines of thought on the subject, one admitting that a select valuation was theoretically correct, the other questioning the advisability of making a select valuation at all, the arguments in favour of the use of an aggregate table being crystallized in Mr. Warner's statement that the logical complement of a select table valuation would be the employment of a rate of interest representing to the nearest penny what had been actually realized during the quinquennium. Considered as a warning against undue striving after exactitude this is unanswerable, but when the calculation of the office reserves is viewed as a preliminary to the declaration of bonus, the conclusion might be drawn that an aggregate table valuation so entirely meets the case as to permit the full distribution of the resulting surplus.

The following considerations appear to demonstrate the advisability of making a select valuation.

It is customary to employ a rate of interest well below the minimum rate likely to prevail in the future, so that the vested portion of the contract is secure and any variations in the rate of interest earned can only affect future profits. The operation of interest is such that in the case of each individual policy the profit from this source is not only practically certain but progressive.

The rate of mortality shown by an aggregate table is, however, an average rate. It is an average not only of the mortality experienced in successive calendar years, but also of the rates experienced in successive policy years; so that in the case of any individual policy the mortality profit disclosed at one stage of the duration may be followed by a loss at a later stage. It follows, therefore, that the mortality profit disclosed where an aggregate table has been used for the purpose of valuation has no easily ascertainable characteristic, such as is shown in the case of interest-profit.

If the data of the British offices whole-life experience be examined by comparing the ungraduated expectations of life at any one attained age, arranged according to duration, with the corresponding expectation based on the aggregate table, that is, if the value of e_x be taken as the base line on which the successive ordinates of $e_{[x-t]+t}$ be set up, for all values of t from 0 to $(x-15)$, it will be found that a curve drawn through the ordinates, commences above the base line, passes below it as the value of t increases and rises above the base line as t approaches the value $(x-15)$.

For example: At attained age 60, the select expectations are greater than the aggregate during the first 10 years of assurance; from duration 10 to duration 29 they are less than the aggregate, but from duration 30 onwards they are again considerably in excess.

Again at attained age 40, the select expectations are less than the aggregate between durations 5 and 11 inclusive, and greater than the aggregate from duration 12 onwards.

This feature is noticeable at all the ages I have tested. If, therefore, the experience of an office is such that the duration of the major portion of the business falls within the middle period, the aggregate rates of mortality used in the valuation may easily be less than the experience rates, and the reserves will not represent the total of the individual reserves with sufficient accuracy.

This is an extreme supposition in the case of an office, but, it is what actually happens in the history of each individual policy, although the effect is obscured by the method of valuation.

There is fortunately no doubt that in the majority of cases this point is thoroughly well recognized and allowed for, but there have been instances where a falling rate of bonus has indicated some hidden weakness. And even where skill and experience succeed in safely conserving the interests of the policy-holder it appears regrettable that reliance should be placed on empirical rules when a more scientific procedure is possible.

Unless a select valuation is made, it is much more difficult to decide how much of the mortality profit has actually been realized and what provision must be made for the future.

Endowment assurances are now generally recognized as forming the most important branch of life assurance. It is therefore proposed to deal with this class alone, to employ a model endowment assurance office, to value by endowment assurance select functions, and to compare the resulting reserves with those obtained by valuation on several of the bases usually employed.

The first portion of the paper will therefore deal with the construction of the necessary tables of endowment assurance mortality functions, and it is my hope that the second part may afford a means of comparing the effect of the employment of one or other of several tables of mortality from the point of view of the declaration of bonus.

After careful consideration it was decided to employ the British Offices endowment assurance combined experience and to graduate the rates of mortality by the graphic method. Diagram I shows the graduated values of the select and ultimate functions, together with the corresponding $O^{[M]}$ and $O^{M(5)}$ curves.

The data are very scanty at both ends of the table, and it seemed reasonable to suppose that the select and ultimate curves would run into the corresponding whole-life assurance curves. They were made to run in at ages 60 and 70 respectively.

The ultimate curve presented some difficulty as the graduated rates are actually in excess of the $O^{M(5)}$ rates at ages 20 to 24, but bearing in mind that the exposed to risk and the deaths only numbered 2,859 and 24 respectively, of which 302 exposed to risk and 2 deaths were under age 15

at date of entry, it appeared advisable to assume an arbitrary rate at those ages, more especially as it seems probable that some at least of these policies were either deferred assurances or pure endowments and therefore not subject to medical examination.

Graduated values of q_x were obtained graphically for duration 0 and for durations 5 and over, the values so obtained being subsequently adjusted. The values for duration 1 were obtained for every fifth age by subsidiary curves, the column being completed by interpolation. Durations 2, 3 and 4 were then inserted by first difference interpolation.

The result as shown in Table I has been to exaggerate the mortality for durations 2, 3 and 4, but it is thought better to avoid a distinct break between duration 4 and the ultimate table, which would have existed if the ungraduated values at duration 4 had been followed.

Selection has only been observed during the first five years in view of the relatively greater magnitude of endowment assurance reserves and the consequent reduction in the death-strain, after that date, as compared with whole-life assurances.

The graduated values of $q_{[x]+t}$, $l_{[x]+t}$, $D_{[x]+t}$ and $N_{[x]+t}$ are given in Tables II to V. It will be noticed that the radix of the $l_{[x]+t}$ column is such that l_{70} is the same as in the $O^{M(5)}$ Table, the $O^{M(5)}$ commutation columns being used for ages 70 and over.

A somewhat severe test of the graduation of the ultimate table is afforded by a comparison of the new values of $\mathbf{a}_{x:\overline{60-x}|}$ with those published by Mr. Ackland, *J.I.A.*, vol. xxxvii, p. 148.

Judged by this standard the annuity values are too high at ages 20 to 24, and also from age 34 to age 49; between ages 24 and 34, however, they are too low. In effect, I have sacrificed some degree of accuracy in order to secure greater regularity in the progression of the rates of mortality.

The only other available comparison is with Mr. Norton's $3\frac{1}{2}$ per-cent premium rates based on certain percentages of the $O^{[M]}$ rates (*J.I.A.*, vol. xlviii, p. 313), which I give in Table VI together with the corresponding $O^{[M]}$ premiums and the premiums derived from the new rates. At ages 20 and 30 the agreement between the two sets is perfect, but in the case of ages 40 and 50 the new rates lie between Mr. Norton's and the $O^{[M]}$.

This obviously follows from the different assumptions as to the rate of mortality on which the two tables are based.

In Tables VII and VIII are given the values of $\mathbf{a}_{[x]:\overline{n}|}$ and $P_{[x]:\overline{n}|}$ for all values of x from 15 to $(74-n)$ for terms of 15, 20, 25 and 30 years.

If these values can be regarded as sufficiently representative we are now able to supplement Mr. Ackland's comparative tables of endowment assurance (death or 60) policy-values, (*J.I.A.*, vol. xxxvii, pp. 166 *et seq.*), by the inclusion of the $E^{[M]}$ 3 per-cent reserves.

The ratios of the $E^{[M]}$ 3 per-cent policy values to the policy-values based on other tables are given in Table IX.

I have included values at $2\frac{1}{2}$ per-cent by the O^M and the O^M and $O^{M(5)}$ Tables as these rates are employed by several offices. The H^M values are considerably lower than the $E^{[M]}$ values at the shorter durations and younger entry-ages. The O^M 3 per-cent Table shows a decided improvement at the younger ages, but the ratios show that the improvement is not very marked at the older ages.

The composite tables, namely: the H^M and $H^{M(5)}$ and the O^M and $O^{M(5)}$ 3 per-cent are greater up to attained age 35, but fall below the $E^{[M]}$ above this point.

The O^M $2\frac{1}{2}$ per-cent and the O^M and $O^{M(5)}$ $2\frac{1}{2}$ per-cent are both shown to be very stringent, when considered as substitutes for the $E^{[M]}$ 3 per-cent Table.

Before leaving this part of the subject, it may be remarked that the rates of mortality have been purposely exaggerated, partly to avoid a charge of undue optimism, and partly in deference to the opinion of those who hold that the low rates result from the greater vitality of the comparatively small number who effected endowment assurances in the early days and cannot be maintained now that this class of assurance has become the vogue. My own opinion, however, is that the low rates will be found to persist. This is based principally on the examination of the experience of nearly 101,000 policies issued since 1910 having 171,712 years of exposure and 488 deaths.

In this experience for the first three policy years of assurance the deaths were 69·4 per-cent, 77·6 per-cent and 79·1 per-cent respectively of those expected under the $E^{[M]}$ Table. These percentages are not to be considered as being representative of the experience of all offices combined, for there is no doubt that the rate of mortality is influenced to a large extent by the term of the endowment assurance, as well as by the number

of years elapsed since date of entry, and an examination of the Board of Trade returns discloses that the average term of assurance is far from being the same in all companies.

This may be due to the different manner in which the companies approach the public. Some publish their premium rates tabulated according to age at maturity, others as payable in a given number of years.

The nature of the business may be influenced by this factor, for it does not seem unreasonable to suppose that a man is more likely to effect an assurance payable at age 60, if it is offered in that form, rather than as an assurance to mature in 40 years. If this is so, then the offices comprising the AT group are more likely to have policies with a longer average term than those which may be termed the IN group.

When the Institute undertakes the compilation of the next mortality experience, we shall have a body of endowment assurance data of such considerable magnitude that it should be possible to subdivide the data according to the original term as well as duration and age at entry. This should undoubtedly be done if the full benefit of the experience is to be realized.

It now remains to describe as briefly as possible the construction of the Model Office.

Dr. Buchanan's model endowment office at once suggested itself, and it was with regret that I felt myself unable to use it as it stands, but dealing as I am with a select valuation it is desirable to value the assurances as at the middle of the policy year. Thus the reserves would not be comparable with Dr. Buchanan's, which are ascertained as at the commencement of the policy year.

In addition to this I thought it would be interesting to show the effect of an entirely different scale of distribution, as regards the number of policies effected for the various maturity ages and endowment terms.

In col. (1) of Table X is given the percentage distribution according to term, of the new business, excluding the 10 year policies, in Dr. Buchanan's model office. It will be noticed that it differs entirely from the scale that I have adopted, which is given in col. (2). The latter may, I think, be considered as fairly representative of an office of the IN type.

The exposed to risk at duration 0 in the British Office New Experience were divided according to these percentages in ten yearly age groups. In view of the method adopted in the

tabulation of the data, these may be taken to represent the survivors in the middle of the first policy year, that is, at the end of the first calendar year, if we assume that the assurances are effected on the 30th June in each calendar year.

The survivors at the end of each successive calendar year were obtained by using Dr. Buchanan's combined rates of death and withdrawal, the result being a slight increase in the number of survivors. The model office thus obtained is given in Table XI.

We now have at our disposal the means of testing the effect of the various tables generally used in the valuation of endowment assurances.

The basis of valuation is invariably chosen with a view to maintaining the rate of bonus. I have, therefore, endeavoured to demonstrate in what manner the bonus declaration of the model office would be affected, according as the smaller reserves of the H^{M} or O^{M} Tables at 3 per-cent or the more stringent reserves of the O^{M} $2\frac{1}{2}$ per-cent or O^{M} and $O^{M(5)}$ $2\frac{1}{2}$ per-cent be employed in place of the $E^{[M]}$ 3 per-cent.

In order to clear the issue, it has been assumed that the office charges premiums sufficient to meet all expenses, to pay the claims as they arise, and to provide a simple reversionary bonus of 30*s*. per-cent per annum, declared quinquennially on the $E^{[M]}$ 3 per-cent basis, with intermediate bonuses at the rate last declared, which in this case remains unchanged. This relieves us of all assumptions as to profit from the various sources except a small amount of that due to mortality and interest. It is, therefore, only assumed that $E^{[M]}$ mortality will be experienced, and that the funds will be invested to yield 4 per-cent free of income-tax.

The valuation basis will affect the profit from the surrender values allowed by the office, in so far as the cash value of the bonus additions is concerned. This will consequently have an effect on the gross profit of the quinquennium, but it is quite impossible to take it into account at this stage and my results will therefore require to be somewhat modified although not to any very great extent. Further reference will be made to this point when we deal with a change in the valuation basis.

As I have already had occasion to extract the exposed to risk at each of the 30 durations, in ten yearly age-groups,

in order to test Mr. A. E. King's method of group valuation, I made use of these data to obtain the total $E^{[M]}$ 3 per-cent pure premiums at each duration, the result has been that the average entry ages, deduced therefrom, vary between small limits.

In passing it may be remarked that Mr. A. E. King's method of group valuation is particularly well adapted to a valuation by select tables, during the period of selection, as the bonus additions at each duration are free from the disturbances due to the merging of different durations that arise in a valuation by an aggregate table.

Table XII contains the valuation results, on the $E^{[M]}$ 3 per-cent basis of the model office as at the end of each quinquennial period. The funds have been ascertained by adding the surplus required to provide a 30*s*. bonus, to the reserve values for the sum assured and existing bonus additions.

Table XIII shows the effect on the divisible surplus if the same office employed the H^M 3 per-cent Table in its valuation.

At the end of the first quinquennium, the reserves are £9,820 less than those required by the standard table, and as the funds remain unaltered the cash surplus is increased from £63,329 to £73,149, which is sufficient to provide a simple reversionary bonus of 1·712 per-cent per annum.

During the next quinquennium the funds are depleted by the payment of the difference between the reversionary bonus of 1·500 and 1·712 in respect of each claim arising by death during the intervaluation period, the total sum paid, accumulated at 4 per-cent per annum, amounts to £1,197, so that instead of holding funds of £1,227,198 the office has £1,226,001 in hand, and can only declare a bonus at the rate of 1·621 per-cent per annum.

By the end of the fourth quinquennium, the rate of bonus is less than 1·500 per-cent, and the fall continues until the end of the ninth quinquennium after when the adverse effect of the excessive bonus allotted to the early policy-holders begins to wear off and the bonus slowly approaches the 30*s*. standard.

The effect of valuing on the more stringent basis of the O^M mortality is shown in Table XIV.

The reserves, although higher than those of the H^M Table, are less than the standard, with the result that the early bonuses are still in excess of 30*s*., but their adverse effect is not so pronounced.

Tables XV and XVI give the result of valuing on the O^{M} $2\frac{1}{2}$ per-cent and O^{M} and $O^{M(5)}$ $2\frac{1}{2}$ per-cent basis. They show that the early policy-holders are penalized in order to provide a too favourable rate of bonus to the later entrants.

If the office ceases to transact new business, the H^{M} and O^{M} 3 per-cent bonuses will continue to fall, while the O^{M} $2\frac{1}{2}$ per-cent and O^{M} and $O^{M(5)}$ $2\frac{1}{2}$ per-cent bonuses will rise, until the office finally closes its doors after payment of the last matured assurance.

We may therefore restrict our further investigations to the H^{M} and O^{M} 3 per-cent valuations. In the former case the net result of 50 years' business is that, instead of holding funds of £3,625,003, the office only holds £3,570,670, the excess payments of bonus having accumulated to £54,333, that is, nearly $1\frac{1}{2}$ per-cent of the funds.

In order to place the office on an $E^{[M]}$ 3 per-cent basis with a 30*s*. bonus it is only necessary to increase the funds by £44,995, comprising £39,040 increased reserve values, and £5,955 which is required to bring the cash surplus of £332,435 up to £338,390 in order to provide a 30*s*. bonus on the $E^{[M]}$ 3 per-cent basis.

The balance of the deficit in the funds, namely, £9,338 represents the additional reserve that would have been required if the full 30*s*. bonus had been declared, and will be gradually extinguished as the policies terminate by death, maturity and surrender.

The present value of the forfeited bonus on future claims by death and maturity is given in col. 5 of Table XVII as at the end of each quinquennium.

At the end of the tenth quinquennium this amounts to £8,528, showing that the present value of the forfeited bonus on future surrenders amounts to £810. Five years later the amounts are £3,779 and £255 respectively, thus the sum of £555 has accrued to the funds of the office, from the termination of policies other than by death and maturity during the quinquennium, this amount is sufficient to provide an additional bonus of ·002 per-cent, which may be considered the maximum amount of disturbance arising from the neglect of this source of profit.

In the early years of the company's existence the higher bonuses then standing on policies terminated by surrender, involve the office in a loss, and thus the fall in the rate of bonus is accentuated at each successive valuation.

The result of valuing by the O^{M} 3 per-cent Table is more satisfactory. The deficit at the end of 50 years amounts to £39,275, of which £6,677 will be recovered from the reduced bonus allotted to the policies then in existence; beyond this the figures do not call for a detailed examination. For the purpose of the investigation it has been assumed that the office in question has distributed all its surplus, but it is evident that results identical with those obtained would follow, if the office were to make profits more than sufficient to declare a 30*s.* bonus on the $E^{[M]}$ 3 per-cent valuation basis, provided such additional surplus remained undistributed.

It is from such a source that the deficit would be made good in the examples given, but even so, the resources of the office would have been reduced to the extent of the deficit and the future policy-holders would suffer correspondingly.

I have not lost sight of the fact that the higher rate of bonus would probably attract new entrants, but this is a doubtful advantage, as it merely defers the period of depression, and when this has arrived and the bonus is no longer attractive the new business will probably fall off with disastrous results.

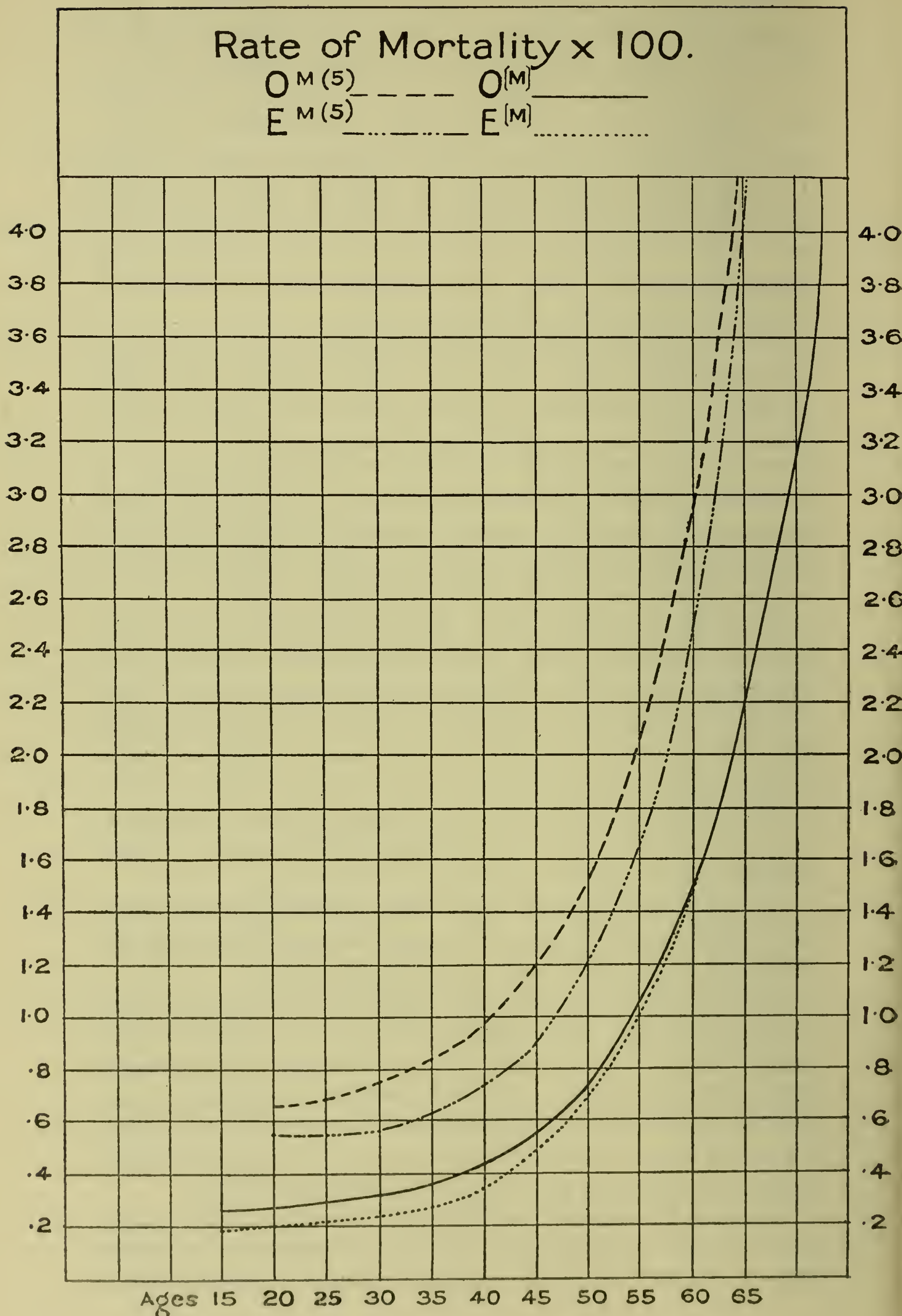
In order to illustrate this I have constructed a model office from the exposed to risk in the British Offices whole-life new experience which may be considered as representing an office with an increasing new business or with an excessively high rate of discontinuance, both of those causes having a similar effect.

The valuation figures are of no intrinsic value in themselves, and I have therefore given the bonus results alone in Table XVIII.

It will be seen that the effect of the altered scale of business is to delay the reduction in the bonus, before it reaches the 30*s.* point and to accelerate the reduction afterwards.

The problem may now be considered from another point of view, namely, what proportion of the divisible surplus can be distributed with safety on the different valuation bases? Here the investigation is simplified, for the funds of the company are not affected by any variation in the rate of bonus, and it is a simple matter to ascertain the percentage of divisible surplus to be carried forward after providing for the 30*s.* bonus.

Table XIX exhibits the position of the office at the end of each quinquennium until it reaches a stationary condition after 30 years. From this table we see that a well-established


Rate of Mortality x 100.
O M (5) O (M)
E M (5) E (M)
4·0
3·8
3·6
3·4
3·2
3·0
2·8
2·6
2·4
2·2
2·0
1·8
1·6
1·4
1·2
1·0
·8
·6
·4
·2
Ages 15 20 25 30 35 40 45 50 55 60 65

office valuing on the H^M basis can safely distribute about 90 per-cent of its surplus, and about 93 per-cent if it values on the O^M 3 per-cent basis. Table XIX, taken in conjunction with Table XIII, enables us to see what would happen if an office valuing by the H^M Table were to declare a 30*s.* bonus during the first 30 years of its existence and then decided to increase the bonus to the full extent possible. If this were done, a bonus at the rate of 1·6 per-cent could be declared at the next valuation, but, as before, this rate would fall at each subsequent valuation until it eventually stands at less than the normal rate of 30*s.*

The foregoing investigation may appear to have resulted in a somewhat obvious conclusion, but the increasing importance of endowment assurances combined with the fact that it has not been the invariable rule for offices to carry forward part of their ascertained surplus, even when valuing on the H^M or O^M 3 per-cent basis, seems to indicate a necessity for an authoritative expression of opinion on the subject, and it is my desire that such may be forthcoming in the course of the discussion on this, admittedly, highly debatable question.

TABLE I.

Comparison of Expected and Actual Deaths.

Age at Entry [x]	0		1		2		3		4		5 and over		Age attained $x+5$
	Ex-pected	Actual	Ex-pected	Actual	Ex-pected	Actual	Ex-pected	Actual	Ex-pected	Actual	Ex-pected	Actual	
20–24	58·8	57	74·8	97	73·9	86	72·3	82	70·6	64	183·0	176	25–29
25–29	83·6	81	108·5	94	107·9	99	106·2	101	103·9	103	460·9	441	30–34
30–34	70·9	86	95·8	76	96·1	89	95·7	90	93·7	86	642·1	613	35–39
35–39	51·4	52	67·6	77	67·4	63	66·8	60	65·5	54	699·1	739	40–44
40–44	35·3	39	43·1	41	42·8	38	42·0	34	41·5	30	702·6	741	45–49
45–49	17·8	18	20·6	10	20·6	25	20·0	18	19·7	11	560·8	542	50–54
50–54	7·3	5	7·9	12	7·8	4	7·6	6	7·3	8	410·8	440	55–59
55–59	2·1	2	2·4	5	2·3	2	2·4	2	2·2	4	...	...	60–64
	327·2	340	420·7	412	418·8	406	413·0	393	404·4	360	3659·3	3692	

TABLE II.

$E^{[M]}$

$q_{[x]+t}$

$[x]$	Year of Assurance 0	1	2	3	4	5 & over	$[x]+5$
15	·00190	·00310	·00368	·00427	·00491	·00548	20
16	192	312	370	428	491	548	21
17	194	314	372	429	492	548	22
18	196	316	374	431	492	548	23
19	198	318	376	433	493	550	24
20	·00200	·00320	·00378	·00435	·00493	·00550	25
21	202	322	379	436	493	550	26
22	204	324	381	437	494	550	27
23	206	326	383	439	496	552	28
24	208	329	386	442	499	555	29
25	·00210	·00333	·00390	·00447	·00503	·00560	30
26	213	337	395	452	510	567	31
27	216	342	401	459	518	576	32
28	219	348	408	468	527	587	33
29	222	355	416	478	539	600	34
30	·00225	·00363	·00426	·00489	·00552	·00615	35
31	229	372	437	502	567	632	36
32	235	382	449	517	584	651	37
33	244	394	464	533	603	672	38
34	254	408	480	552	623	695	39
35	·00266	·00424	·00498	·00572	·00646	·00720	40
36	280	442	518	595	671	747	41
37	296	462	541	620	698	777	42
38	314	484	566	648	730	812	43
39	334	508	594	680	766	852	44
40	·00356	534	·00625	·00716	·00807	·00897	45
41	380	562	659	754	850	947	46
42	406	593	696	798	900	1002	47
43	434	627	736	845	954	1062	48
44	465	664	780	896	1012	1127	49
45	·00500	·00704	·00828	·00951	·01074	·01197	50
46	538	748	880	1011	1143	1274	51
47	580	796	937	1077	1218	1358	52
48	626	848	998	1149	1299	1449	53
49	676	905	1066	1227	1388	1549	54
50	·00726	·00968	·01141	·01314	·01487	·01660	55
51	781	1039	1226	1412	1598	1784	56
52	830	1120	1321	1522	1723	1924	57
53	899	1213	1431	1649	1867	2084	58
54	965	1320	1558	1795	2032	2269	59
55	·01035	·01443	·01702	·01961	·02220	·02479	60
56	1113	1580	1865	2150	2435	2719	61
57	1197	1729	2044	2359	2674	2989	62
58	1289	1888	2240	2591	2943	3294	63
59	1389	2055	2450	2845	3240	3634	64
60	·01499	·02228	·02672	·03116	·03560	·04004	65
61	...	...	...	...	...	4404	66
62	...	...	...	...	...	4834	67
63	...	...	...	...	...	5288	68
64	...	...	...	...	...	5748	69
65	...	...	...	...	...	·06219	70

TABLE III.

$E^{[M]}$

$l_{[x]+t}$

$[x]$	Year of Assurance: 0	1	2	3	4	5 & over	$[x]+5$
15	89899	89728	89450	89121	88740	88304	20
16	89411	89239	88961	88632	88253	87820	21
17	88930	88757	88478	88149	87771	87339	22
18	88448	88275	87996	87667	87289	86860	23
19	87972	87798	87519	87190	86812	86384	24
20	87495	87320	87041	86712	86335	85909	25
21	87020	86844	86564	86236	85860	85437	26
22	86549	86372	86092	85764	85389	84967	27
23	86080	85903	85623	85295	84921	84500	28
24	85618	85440	85159	84830	84455	84034	29
25	85159	84980	84697	84367	83990	83568	30
26	84703	84523	84238	83905	83526	83100	31
27	84248	84066	83778	83442	83059	82629	32
28	83791	83607	83316	82976	82588	82153	33
29	83333	83148	82853	82508	82114	81671	34
30	82869	82683	82383	82033	81632	81181	35
31	82404	82215	81909	81551	81142	80682	36
32	81933	81740	81428	81062	80643	80172	37
33	81457	81258	80938	80562	80133	79650	38
34	80975	80769	80439	80053	79611	79115	39
35	80483	80269	79929	79531	79076	78565	40
36	79979	79755	79402	78991	78526	77999	41
37	79474	79239	78873	78446	77960	77416	42
38	78956	78708	78327	77884	77379	76814	43
39	78425	78163	77766	77304	76778	76190	44
40	77877	77600	77186	76704	76155	75541	45
41	77311	77017	76584	76079	75505	74863	46
42	76722	76411	75958	75429	74827	74154	47
43	76109	75779	75304	74750	74118	73411	48
44	75469	75118	74619	74037	73374	72631	49
45	74799	74425	73901	73289	72592	71812	50
46	74099	73700	73149	72505	71772	70952	51
47	73369	72943	72362	71684	70912	70048	52
48	72600	72146	71534	70820	70006	69097	53
49	71795	71310	70665	69912	69054	68096	54
50	70952	70437	69755	68959	68053	67041	55
51	70071	69524	68802	67958	66999	65928	56
52	69143	68569	67801	66905	65887	64752	57
53	68187	67574	66754	65799	64714	63506	58
54	67182	66534	65656	64633	63473	62183	59
55	66123	65439	64495	63397	62152	60772	60
56	64997	64274	63259	62079	60744	59265	61
57	63789	63025	61935	60669	59238	57654	62
58	62486	61681	60515	59159	57626	55931	63
59	61068	60220	58982	57537	55900	54089	64
60	59514	58622	57316	55785	54047	52123	65
61	...	...	...	...	...	50036	66
62	...	...	...	...	...	47833	67
63	...	...	...	...	...	45521	68
64	...	...	...	...	...	43114	69
65	...	...	...	...	...	40636	70

TABLE IV.

$E^{[M]}$ $D_{[x]+t}$ 3 *per-cent.*

$[x]$	Year of Assurance 0	1	2	3	4	5 & over	$[x]+5$
15	57703	55916	54119	52349	50608	48892	20
16	55718	53991	52255	50546	48864	47208	21
17	53804	52135	50458	48806	47181	45581	22
18	51953	50342	48722	47125	45555	44011	23
19	50170	48612	47046	45504	43987	42495	24
20	48444	46939	45426	43936	42471	41031	25
21	46778	45323	43861	42422	41008	39616	26
22	45169	43764	42351	40962	39594	38251	27
23	43616	42258	40894	39550	38231	36933	28
24	42118	40807	39487	38190	36914	35660	29
25	40673	39404	38130	36875	35641	34429	30
26	39276	38051	36819	35605	34412	33239	31
27	37928	36744	35551	34377	33223	32088	32
28	36623	35479	34325	33190	32072	30974	33
29	35362	34256	33140	32041	30959	29895	34
30	34141	33072	31993	30929	29881	28850	35
31	32961	31927	30882	29851	28836	27838	36
32	31818	30818	29806	28808	27824	26856	37
33	30712	29744	28764	27796	26843	25905	38
34	29640	28704	27754	26816	25892	24981	39
35	28602	27695	26775	25866	24968	24085	40
36	27595	26716	25824	24941	24073	23215	41
37	26622	25771	24904	24048	23203	22370	42
38	25679	24852	24012	23181	22359	21549	43
39	24763	23962	23145	22338	21539	20752	44
40	23874	23096	22304	21519	20742	19976	45
41	23010	22255	21485	20722	19967	19220	46
42	22170	21436	20689	19946	19211	18484	47
43	21352	20640	19913	19191	18475	17765	48
44	20555	19864	19158	18454	17757	17065	49
45	19780	19108	18421	17736	17055	16381	50
46	19024	18371	17702	17035	16372	15713	51
47	18288	17652	17001	16352	15704	15061	52
48	17569	16951	16318	15684	15052	14424	53
49	16868	16267	15649	15032	14415	13801	54
50	16185	15599	14998	14395	13792	13192	55
51	15518	14948	14362	13773	13183	12595	56
52	14866	14314	13741	13165	12587	12010	57
53	14234	13695	13135	12570	12003	11436	58
54	13615	13092	12542	11987	11430	10871	59
55	13011	12501	11961	11416	10866	10315	60
56	12417	11921	11391	10853	10310	9766·3	61
57	11830	11349	10828	10297	9761·8	9224·1	62
58	11252	10784	10271	9748·8	9219·6	8687·8	63
59	10677	10221	9719·7	9205·3	8682·9	8157·2	64
60	10101	9660·3	9170·0	8665·1	8150·8	7631·3	65
61	...	...	...	...	...	7112·6	66
62	...	...	...	...	...	6601·4	67
63	...	...	...	...	...	6099·4	68
64	...	...	...	...	...	5608·7	69
65	...	...	...	...	...	5132·2	70

TABLE V.

$E^{[M]}$ $\mathbb{N}_{[x]+t}$ 3 *per-cent.*

$[x]$	Year of Assurance 0	1	2	3	4	5 & over	$[x]+5$
15	1444667	1386964	1331048	1276929	1224580	1173972	20
16	1386454	1330736	1276745	1224490	1173949	1125080	21
17	1330256	1276452	1224317	1173859	1125053	1077872	22
18	1275988	1224035	1173693	1124971	1077846	1032291	23
19	1223600	1173429	1124817	1077771	1032267	988280	24
20	1173001	1124557	1077618	1032192	988256	945785	25
21	1124146	1077368	1032045	988184	945762	904754	26
22	1076978	1031809	988045	945694	904732	865138	27
23	1031436	987820	945562	904668	865118	826887	28
24	987470	945352	904545	865058	826868	789954	29
25	945017	904344	864940	826810	789935	754294	30
26	904028	864752	826701	789882	754277	719865	31
27	864449	826521	789777	754226	719849	686626	32
28	826227	789604	754125	719800	686610	654538	33
29	789322	753960	719704	686564	654523	623564	34
30	753685	719544	686472	654479	623550	593669	35
31	719276	686315	654388	623506	593655	564819	36
32	686055	654237	623419	593613	564805	536981	37
33	653984	623272	593528	564764	536968	510125	38
34	623026	593386	564682	536928	510112	484220	39
35	593145	564543	536848	510073	484207	459239	40
36	564303	536708	509992	484168	459227	435154	41
37	536487	509865	484094	459190	435142	411939	42
38	509652	483973	459121	435109	411928	389569	43
39	483767	459004	435042	411897	389559	368020	44
40	458803	434929	411833	389529	368010	347268	45
41	434731	411721	389466	367981	347259	327292	46
42	411524	389354	367918	347229	327283	308072	47
43	389159	367807	347167	327254	308063	289588	48
44	367611	347056	327192	308034	289580	271823	49
45	346858	327078	307970	289549	271813	254758	50
46	326881	307857	289486	271784	254749	238377	51
47	307661	289373	271721	254720	238368	222664	52
48	289177	271608	254657	238339	222655	207603	53
49	271410	254542	238275	222626	207594	193179	54
50	254347	238162	222563	207565	193170	179378	55
51	237970	222452	207504	193142	179369	166186	56
52	222264	207398	193084	179343	166178	153591	57
53	207218	192984	179289	166154	153584	141581	58
54	192811	179196	166104	153562	141575	130145	59
55	179029	166018	153517	141556	130140	119274	60
56	165851	153434	141513	130122	119269	108959	61
57	153258	141428	130079	119251	108954	99193	62
58	141244	129992	119208	108937	99188	89969	63
59	129787	119110	108889	99169	89964	81281	64
60	118870	108769	99109	89939	81274	73124	65
61	...	...	...	...	...	65492	66
62	...	...	...	...	...	58380	67
63	...	...	...	...	...	51778	68
64	...	...	...	...	...	45679	69
65	...	...	...	...	...	40070	70

TABLE VI.

Endowment Assurance Premium Rates. $3\frac{1}{2}$ *per-cent.*

Term of Assurance	Entry Age 20 E[M]	20 O[M]	20 Approximate	30 E[M]	30 O[M]	30 Approximate	40 E[M]	40 O[M]	40 Approximate	50 E[M]	50 O[M]	50 Approximate
10	8·43	8·47	8·42	8·45	8·52	8·45	8·56	8·63	8·53	8·84	8·89	8·73
15	5·24	5·30	5·24	5·28	5·36	5·27	5·42	5·52	5·39	5·82	5·91	5·68
20	3·68	3·76	3·68	3·74	3·84	3·74	3·93	4·05	3·90	4·46	4·57	4·29
25	2·78	2·87	2·78	2·86	2·98	2·86	3·11	3·25	3·07	...	...	...
30	2·20	2·30	2·21	2·31	2·45	2·32	2·63	2·79	2·59	...	...	...
35	1·82	1·93	1·83	1·96	2·11	1·97	...	...	...	...	...	...
40	1·55	1·67	1·57	1·74	1·90	1·75	...	...	...	...	...	...
45	1·37	1·49	1·39	...	...	...	...	...	...	...	...	...
50	1·25	1·39	1·27	...	...	...		...	...	...	...	...

TABLE VII.

$E^{[M]}$ $\mathbf{a}_{[x]:\overline{n}|}$ 3 *per-cent.*

x	n = 15	20	25	30
15	11·964	14·748	17·077	19·018
16	11·964	14·747	17·074	19·010
17	11·963	14·744	17·068	18·998
18	11·962	14·741	17·062	18·986
19	11·960	14·737	17·053	18·971
20	11·959	14·733	17·045	18·954
21	11·957	14·729	17·035	18·936
22	11·955	14·723	17·023	18·914
23	11·952	14·716	17·008	18·888
24	11·949	14·708	16·992	18·859
25	11·943	14·696	16 971	18·824
26	11·938	14·684	16·948	18·786
27	11·931	14·669	16·921	18·742
28	11·923	14·653	16·892	18·694
29	11·914	14·634	16·858	18·641
30	11·904	14·614	16·821	18·582
31	11·892	14·590	16·780	18·516
32	11·879	14·563	16·734	18·444
33	11·865	14·534	16·684	18·364
34	11·849	14·502	16·629	18·277
35	11·831	14·467	16·568	18·182
36	11·811	14·427	16·501	18·076
37	11·788	14·383	16·426	17·959
38	11·762	14·334	16·343	17·831
39	11·734	14·280	16·254	17·691
40	11·704	14·222	16·155	17·539
41	11·671	14·158	16·047	17·375
42	11·634	14·088	15·929	17·197
43	11·595	14·012	15·801	17·007
44	11·553	13·930	15·662	16·803
45	11·506	13·839	15 510	16·584
46	11·455	13·740	15·346	...
47	11·399	13·630	15·168	...
48	11·339	13·512	14·978	...
49	11·272	13·382	14·773	...
50	11·197	13·239	14·551	...
51	11·115	13·084	...	...
52	11·024	12·915	...	...
53	10·920	12·729	...	...
54	10·807	12·529	...	...
55	10·680	12·313	...	...
56	10·543	...	...	...
57	10·397	...	...	...
58	10·239	...	...	...
59	10·074	...	...	...
60	9·905	...	...	...

TABLE VIII.

$E^{[M]}$ $P_{[x]:\overline{n}|}$ 3 *per-cent.*

x	n = 15	20	25	30
15	·05446	·03868	·02944	·02346
16	5446	3868	2945	2348
17	5447	3870	2947	2352
18	5448	3871	2948	2354
19	5449	3872	2951	2359
20	·05450	·03874	·02954	·02363
21	5451	3876	2957	2368
22	5452	3879	2962	2375
23	5455	3882	2968	2382
24	5457	3886	2972	2389
25	·05461	03890	·02980	·02400
26	5464	3897	2989	2410
27	5469	3904	2998	2423
28	5475	3911	3007	2437
29	5481	3921	3019	2452
30	·05488	·03930	·03033	·02469
31	5497	3941	3047	2488
32	5506	3954	3063	2509
33	5515	3968	3081	2533
34	5527	3983	3100	2559
35	·05539	·03999	·03123	·02587
36	5554	4018	3148	2618
37	5570	4039	3176	2655
38	5590	4064	3206	2696
39	5610	4090	3239	2740
40	·05631	·04119	·03277	·02789
41	5655	4151	3319	2842
42	5683	4186	3365	2903
43	5712	4224	3416	2968
44	5743	4266	3472	3039
45	·05779	·04313	·03535	·03117
46	5817	4365	3604	...
47	5860	4424	3681	...
48	5907	4488	3764	...
49	5958	4560	3856	...
50	·06018	·04641	·03959	...
51	6084	4731	...	...
52	6159	4832	...	...
53	6245	4944	...	...
54	6341	5070	...	...
55	·06451	·05209	...	...
56	6572	...	...	...
57	6706	...	...	...
58	6854	...	...	...
59	7014	...	...	...
60	·07183	...	...	...

Table IX.

Ratios of Policy-Values. Endowment Assurances, payable at 60 or previous death.

$E^{[M]}$ 3 *per-cent.* : H^M 3 *per-cent.*

Age at Entry (x)	Duration (n)						
	5	10	15	20	25	30	35
20	108·6	102·7	102·0	101·9	101·6	101·3	100·8
25	106·5	103·4	102·8	102·1	101·6	100·9	...
30	107·5	104·3	102·9	102·0	101·1	...	...
35	107·2	103·6	102·3	101·2	...	...	...
40	105·1	102·6	101·3	...	...	...	...
45	103·9	101·6	...	...	...	...	...
50	102·2	...	...	...	...	...	...

$E^{[M]}$ 3 *per-cent.* : O^M 3 *per-cent.*

Age at Entry (x)	Duration (n)						
	5	10	15	20	25	30	35
20	100·3	97·0	97·6	98·6	99·4	99·9	100·2
25	102·2	99·8	99·9	100·3	100·5	100·4	...
30	103·9	101·6	101·2	101·0	100·6	...	...
35	104·6	102·3	101·5	100·8	...	...	...
40	104·3	102·1	101·0	...	...	...	...
45	103·3	101·3	...	...	...	...	...
50	101·9	...	...	...	...	...	...

$E^{[M]}$ 3 *per-cent.* : H^M *and* $H^{M(5)}$ 3 *per-cent.*

Age at Entry (x)	Duration (n)						
	5	10	15	20	25	30	35
20	83·3	95·9	98·8	100·6	100·9	101·1	100·7
25	94·1	98·9	101·0	101·3	101·2	100·8	...
30	99·1	101·9	101·8	101·5	101·0	...	...
35	102·5	102·2	101·8	101·1	...	...	...
40	102·4	101·9	101·2	...	...	...	...
45	102·5	101·4	...	...	...	...	...
50	101·9	...	...	...	...	...	...

TABLE IX—*continued*.

Ratios of Policy-Values. *Endowment Assurances, payable at 60 or previous death.*

$E^{[M]}$ 3 *per-cent.* : O^{M} *and* $O^{M(5)}$ 3 *per-cent.*

Age at Entry (x)	Duration (n)						
	5	10	15	20	25	30	35
20	83·7	92·0	95·8	97·9	99·1	99·8	100·1
25	92·6	97·2	99·1	100·0	100·3	100·4	...
30	98·9	100·4	100·8	100·8	100·6	...	...
35	102·2	101·7	101·3	100·8	...	...	...
40	103·2	101·8	101·0	...	...	...	...
45	102·9	101·2	...	...	...	...	...
50	101·9	...	...	...	...	...	...

$E^{[M]}$ 3 *per-cent.* : O^{M} $2\frac{1}{2}$ *per-cent.*

Age at Entry (x)	Duration (n)						
	5	10	15	20	25	30	35
20	91·7	89·7	91·1	93·2	95·1	96·9	98·5
25	94·6	93·4	94·6	96·1	97·5	98·8	...
30	97·4	96·3	97·1	98·1	99·1	...	...
35	99·3	98·3	98·7	99·3	...	...	...
40	100·3	99·3	99·6	...	...	...	...
45	100·6	99·9	...	...	...	...	...
50	100·6	...	...	...	...	...	...

$E^{[M]}$ 3 *per-cent.* : O^{M} *and* $O^{M(5)}$ $2\frac{1}{2}$ *per-cent.*

Age at Entry (x)	Duration (n)						
	5	10	15	20	25	30	35
20	77·4	85·3	89·6	92·5	94·9	96·8	98·5
25	86·2	91·1	93·8	95·8	97·4	98·8	...
30	92·9	95·1	96·7	98·0	99·1	...	...
35	97·1	97·7	98·5	99·3	...	...	...
40	99·2	99·1	99·5	...	...	...	...
45	100·2	99·8	...	...	...	...	...
50	100·5	...	...	...	...	...	...

TABLE X.

Percentage Distribution of New Business.

Col. (1) Dr. BUCHANAN'S Model Office.
Col. (2) Proposed ,, ,,

Age at Entry	Term of Assurance													
	15		20		25		30		35		40		...	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
15–24	3	30	6	60	8	9	33	1	26	...	24	...	100	100
25–34	4	30	23	61	29	7	38	2	6	...	...	...	100	100
35–44	16	46	64	47	18	6	2	1	...	...	...	...	100	100
45–54	85	82	15	17	...	1	...	...	...	...	...	...	100	100
55–64	...	100	...	...	...	...	...	...	...	...	...	...	...	100

TABLE XI.

Model Office for Endowment Assurances.

Sums Assured existing at the end of each calendar year, i.e., in the middle of each Policy Year.

Duration	Endowment Term 15 Years	20 Years	25 Years	30 Years
0	199980	241512	29508	6241
1	195368	234668	28594	6026
2	184902	218460	26316	5494
3	178156	207724	24858	5159
4	173311	199837	23797	4915
5	169517	193693	22989	4733
6	166537	188532	22320	4587
7	163396	183745	21719	4458
8	160310	179376	21206	4350
9	157202	175101	20733	4252
10	154769	171205	20295	4164
11	151643	167100	19861	4077
12	148557	163018	19427	3992
13	145983	159130	19010	3909
14	143116	155210	18592	3828
15	...	151049	18168	3748
16	...	146991	17755	3667
17	...	142997	17335	3587
18	...	139534	16926	3507
19	...	136116	16511	3428
20	...	...	16060	3344
21	...	...	15643	3264
22	...	...	15227	3184
23	...	...	14876	3105
24	...	...	14526	3025
25	...	...	...	2942
26	...	...	...	2860
27	...	...	...	2782
28	...	...	...	2710
29	...	...	...	2644

TABLE XII.

$E^{[M]}$ 3 *per-cent. Quinquennial Distribution of Profits.*

Age of Office	Assurance Fund	Reserves: Sum Assured	Bonus Addition	Total Reserve	Surplus	Value of R. B. of 1 per-cent per annum on original Sum Assured	Rate of Bonus per annum
5	352852	289523	...	289523	63329	42219	1·500
10	1227198	996900	62735	1059635	167563	111709	1·500
15	2625264	2115474	234830	2350304	274960	183307	1·500
20	3501280	2789228	380925	3170153	331127	220751	1·500
25	3607253	2866254	403581	3269835	337418	224945	1·500
30	3625003	2878560	408053	3286613	338390	225593	1·500

Table XIII.

Quinquennial Distribution of Profits. *Intermediate Bonus at rate last declared.*

H^M 3 *per-cent.* N.B.—Negative quantities, Italic type H^M 3 *per-cent.*

Age of Office	Assurance Fund on $E^{[M]}$ 3 per-cent basis with Bonus addition at rate of 30*s.* per-cent	Profit or Loss arising from variation of Bonus from a level rate of 30*s.* per-cent			Adjusted Assurance Fund (2) + (5)	Reserves			Surplus (6) − (9)	Value of R. B. of 1 per-cent per annum on original Sum Assured	Rate of Bonus per annum	Rate of Bonus per quin-quennium
		During current quin-quennium	Previous quinquennia accumu-lated at 4 per-cent	Total (3) + (4)		Sum Assured	Bonus addition	Total Reserve				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
5	352852	...	...	...	352852	279703	...	279703	73149	42733	1·712	8·560
10	1227198	*1197*	...	*1197*	1226001	971459	71959	1043418	182583	112617	1·621	8·105
15	2625264	*2303*	*1456*	*3759*	2621505	2079507	258386	2337893	283612	184357	1·538	7·690
20	3501280	*14348*	*4573*	*18921*	3482359	2749909	401935	3151844	330515	221828	1·490	7·450
25	3607253	*14757*	*23020*	*37777*	3569476	2826501	409776	3236277	333199	226026	1·474	7·370
30	3625003	*1778*	*45961*	*47739*	3577264	2838768	405415	3244183	333081	226675	1·469	7·345
35	3625003	6154	*58082*	*51928*	3573075	2838768	401654	3240422	332653	226675	1·468	7·340
40	3625003	9588	*63178*	*53590*	3571413	2838768	400447	3239215	332198	226675	1·466	7·330
45	3625003	10994	*65200*	*54206*	3570797	2838768	399843	3238611	332186	226675	1·465	7·325
50	3625003	11617	*65950*	*54333*	3570670	2838768	399467	3238235	332435	226675	1·467	7·335

TABLE XIV.

Quinquennial Distribution of Profits. *Intermediate Bonus at rate last declared.*

O^M 3 *per-cent.* N.B.—Negative quantities, Italic type. O^M 3 *per-cent.*

Age of Office	Assurance Fund on E[M] 3 per-cent basis with Bonus addition at rate of 30*s.* per-cent	Profit or Loss arising from variation of Bonus from a level rate of 30*s.* per-cent: During current quin-quennium	Profit or Loss …: Previous quinquennia accumu-lated at 4 per-cent	Profit or Loss …: Total (3) + (4)	Adjusted Assurance Fund (2) + (5)	Reserves: Sum Assured	Reserves: Bonus addition	Reserves: Total Reserve	Surplus (6) − (9)	Value of R. B. of 1 per-cent per annum on original Sum Assured	Rate of Bonus per annum	Rate of Bonus per quin-quennium
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
5	352852	...	...	...	352852	281515	...	281515	71337	42597	1·675	8·375
10	1227198	*988*	...	*988*	1226210	977811	70320	1048131	178079	112382	1·585	7·925
15	2625264	*1698*	*1202*	*2900*	2622364	2089116	252529	2341645	280719	184089	1·525	7·625
20	3501280	*10629*	*3528*	*14157*	3487123	2760754	396056	3156810	330313	221554	1·491	7·455
25	3607253	*10558*	*17224*	*27782*	3579471	2837506	407579	3245085	334386	225751	1·481	7·405
30	3625003	*911*	*33801*	*34712*	3590291	2849788	405917	3255705	334586	226400	1·478	7·390
35	3625003	4599	*42232*	*37633*	3587370	2849788	403438	3253226	334144	226400	1·476	7·380
40	3625003	7100	*45786*	*38686*	3586317	2849788	402465	3252253	334064	226400	1·476	7·380
45	3625003	7922	*47067*	*39145*	3585858	2849788	402167	3251955	333903	226400	1·475	7·375
50	3625003	8351	*47626*	*39275*	3585728	2849788	401943	3251731	333997	226400	1·475	7·375

Table XV.

Quinquennial Distribution of Profits. *Intermediate Bonus at rate last declared.*

O^{M} $2\frac{1}{2}$ *per-cent.* N.B.—Negative quantities, Italic type. O^{M} $2\frac{1}{2}$ *per-cent.*

Age of Office	Assurance Fund on $E^{[M]}$ 3 per-cent basis with Bonus addition at rate of 30s. per-cent	Profit or Loss arising from variation of Bonus from a level rate of 30s. per-cent			Adjusted Assurance Fund (2) + (5)	Reserves			Surplus (6) − (9)	Value of R. B. of 1 per-cent per annum on original Sum Assured	Rate of Bonus per annum	Rate of Bonus per quinquennium
		During current quinquennium	Previous quinquennia accumulated at 4 per-cent	Total (3) + (4)		Sum Assured	Bonus addition	Total Reserve				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
5	352852	...	...	...	352852	292535	...	292535	60317	45476	1·326	6·630
10	1227198	982	...	982	1228180	1008221	58190	1066411	161769	118632	1·364	6·820
15	2625264	2391	1195	3586	2628850	2135899	218686	2354585	274265	192158	1·427	7·135
20	3501280	16916	4363	21279	3522559	2813912	364099	3178011	344548	230181	1·497	7·485
25	3607253	19689	25889	45578	3652831	2891455	402675	3294130	358701	234443	1·530	7·650
30	3625003	5100	55452	60552	3685555	2903831	420128	3323959	361596	235099	1·538	7·690
35	3625003	*5622*	73671	68049	3693052	2903831	426441	3330272	362780	235099	1·543	7·715
40	3625003	*11671*	82792	71121	3696124	2903831	429102	3332933	363191	235099	1·545	7·725
45	3625003	*14204*	86529	72325	3697328	2903831	430214	3334045	363283	235099	1·545	7·725
50	3625003	*15111*	87994	72883	3697886	2903831	430565	3334396	363490	235099	1·546	7·730

TABLE XVI.

Quinquennial Distribution of Profits. *Intermediate Bonus at rate last declared.*

O^{M} *and* $O^{M(5)}$ $2\frac{1}{2}$ *per-cent.* N.B.—Negative quantities, Italic type. O^{M} *and* $O^{M(5)}$ $2\frac{1}{2}$ *per-cent.*

Age of Office	Assurance Fund on $E^{[M]}$ 3 per-cent basis with Bonus addition at rate of 30*s.* per-cent	Profit or Loss arising from variation of Bonus from a level rate of 30*s.* per-cent			Adjusted Assurance Fund (2)+(5)	Reserves			Surplus (6)−(9)	Value of R. B. of 1 per-cent per annum on original Sum Assured	Rate of Bonus per annum	Rate of Bonus per quinquennium
		During current quinquennium	Previous quinquennia accumulated at 4 per-cent	Total (3)+(4)		Sum Assured	Bonus addition	Total Reserve				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
5	352852	...	...	...	352852	292535	...	292535	60317	45476	1·326	6·630
10	1227198	982	...	982	1228180	1011035	58225	1069260	158920	118682	1·339	6·695
15	2625264	2731	1195	3926	2629190	2139489	215814	2355303	273887	192222	1·425	7·125
20	3501280	18243	4777	23020	3524300	2817617	361801	3179418	344882	230247	1·498	7·490
25	3607253	21359	28007	49357	3656610	2895167	402057	3297224	359386	234509	1·533	7·665
30	3625003	5398	60050	65448	3690451	2907543	420634	3328177	362274	235165	1·541	7·705
35	3625003	*6276*	79627	73351	3698354	2907543	427268	3334811	363543	235165	1·546	7·730
40	3625003	*12618*	89242	76624	3701627	2907543	430004	3337547	364080	235165	1·548	7·740
45	3625003	*15230*	93225	77995	3702998	2907543	431112	3338655	364343	235165	1·549	7·745
50	3625003	*16296*	94893	78597	3703600	2907543	431636	3339179	364421	235165	1·550	7·750

Table XVII.

Quinquennial Distribution of Profits. Intermediate Bonus at rate last declared.

Valuation Basis altered from H^{M} 3 *per-cent to* $E^{[M]}$ 3 *per-cent.*

N.B.—Negative quantities, Italic type.

Age of Office	Assurance Fund on $E^{[M]}$ 3 per-cent basis with Bonus addition at rate of 30*s.* per-cent	Difference arising from variation of Bonus from a level rate of 30*s.* per-cent (Deaths and Matured)			Value of forfeited Bonus on all Policies	Adjusted Assurance Fund (2) + (6)	Reserves			Surplus (7) − (10)	Value of R. B. of 1 per-cent per annum on original Sum Assured	Rate of Bonus per annum	Rate of Bonus per quinquennium
		During current quinquennium	Previous quinquennia accumulated at 4 per-cent	Total (3) + (4)			Sum Assured	Bonus addition	Total Reserve				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
50	3625003	...	...	*8528*	*9338*	3615665	2878560	398715	3277275	338390	225593	1·500	7·500
55	3625003	6596	*10375*	*3779*	*4033*	3620970	2878560	404020	3282580	338390	225593	1·500	7·500
60	3625003	3564	*4598*	*1034*	*1078*	3623925	2878560	406975	3285535	338390	225593	1·500	7·500
65	3625003	1137	*1258*	*121*	*124*	3624879	2878560	407929	3286489	338390	225593	1·500	7·500
70	3625003	134	*147*	*13*	*14*	3624989	2878560	408039	3286599	338390	225593	1·500	7·500
75	3625003	16	*16*	...	...	3625003	2878560	408053	3286613	338390	225593	1·500	7·500
Valuation Basis altered from O^{M} 3 *per-cent to* $E^{[M]}$ 3 *per-cent.*													
50	3625003	...	...	*6098*	*6677*	3618326	2878560	401376	3279936	338390	225593	1·500	7·500
55	3625003	4724	*7419*	*2695*	*2877*	3622126	2878560	405176	3283736	338390	225593	1·500	7·500
60	3625003	2541	*3279*	*738*	*769*	3624234	2878560	407284	3285844	338390	225593	1·500	7·500
65	3625003	812	*898*	*86*	*89*	3624914	2878560	407964	3286524	338390	225593	1·500	7·500
70	3625003	95	*105*	*10*	*10*	3624993	2878560	408043	3286603	338390	225593	1·500	7·500
75	3625003	12	*12*	...	...	3625003	2878560	408053	3286613	338390	225593	1·500	7·500

TABLE XVIII.

Showing the effect on the comparative Rates of Bonus of an increasing rate of New Business.

Age of Office	Valuation Basis				
	$E^{[M]}$ 3 %	H^{M} 3 %	O^{M} 3 %	O^{M} $2\frac{1}{2}$ %	O^{M} & $O^{M(5)}$ $2\frac{1}{2}$ %
5	1·500	1·711	1·674	1·324	1·324
10	1·500	1·629	1·593	1·362	1·340
15	1·500	1·561	1·543	1·411	1·405
20	1·500	1·520	1·514	1·465	1·463
25	1·500	1·489	1·491	1·507	1·507
30	1·500	1·469	1·476	1·531	1·534
35	1·500	1·455	1·467	1·550	1·554
40	1·500	1·447	1·460	1·563	1·568
45	1·500	1·439	1·455	1·572	1·579
50	1·500	1·436	1·453	1·578	1·585

TABLE XIX.

Age of Office	Assurance Fund on $E^{[M]}$ 3 per-cent basis with Bonus at 30s. per-cent	Valuation Basis H^M 3 per-cent					Valuation Basis O^M 3 per-cent				
		Total Reserve including R. B. of 30s. per-cent per annum	Surplus (2)−(3)	Value of R. B. of 30s. per-cent per annum	Balance of Surplus carried forward	Percentage of Total Surplus carried forward	Total Reserve including R. B. of 30s. per-cent per annum	Surplus (2)−(8)	Value of R. B. of 30s. per-cent per annum	Balance of Surplus carried forward	Percentage of Total Surplus carried forward
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
5	352852	279703	73149	64099	9050	12·4	281515	71337	63895	7442	10·4
10	1227198	1034507	192691	168926	23765	12·3	1040784	186414	168573	17841	9·6
15	2625264	2314968	310296	276535	33761	10·9	2324424	300840	276134	24706	8·2
20	3501280	3131562	369718	332742	36976	10·0	3142236	359044	332331	26713	7·4
25	3607253	3230838	376415	339039	37376	9·9	3241650	365603	338626	26977	7·4
30	3625003	3247590	377413	340013	37400	9·9	3258417	366586	339600	26986	7·4

Abstract of the Discussion.

Mr. H. H. Tayler said that in the discussion of the papers by Messrs. King, Ackland and Diver on methods of valuing whole-life assurances by select tables, the principle of valuation by select tables did not receive unqualified support and the subsequent practice of offices showed few, if any, examples of it. The effect of employing select tables would be relatively less in the case of endowment assurances than in that of whole-life assurances.

In his view, the main result of the paper was to be found in Table XIV, col. 12, which showed that if the experienced mortality corresponded with the $E^{(M)}$ select, and valuation was made by the O^{M} Table and the resulting profit divided, bonuses would be declared first higher than, but gradually falling below, the bonuses which would have resulted from the adoption of the experienced mortality as the basis for valuation. This result emphasized the equivocal character of what was called mortality profit, and he was inclined to think that it was necessary to introduce into one's thoughts a conception of mortality profit less simple than the crude difference of expected and actual strain and more truly expressing the financial advantage or disadvantage to an office of any departure from its expected mortality. The death claims might be postponed but they could not be avoided, and the first inference to be made from the fact that the actual claims or deaths were less than the expected was that the age distribution of deaths was altered in the direction of postponement. Having regard to the nature of the business which formed the basis of the British Offices' endowment assurance experience, they could not assume that present day mortality among endowment assurances—which were now effected for shorter terms and formed not a small group but a large proportion of the whole business—would correspond with the $E^{(M)}$.

From the attitude of actuaries to the valuation of whole-life insurance by select tables it appeared that there was a reluctance to employ enough valuation groups to make an exact valuation, and it was probable that that would stand in the way of select valuation both for whole-life and endowment assurances. He thought, however, it was worth re-examination whether the grouping usually adopted was on the whole most advantageous. From the fact that in industrial assurance valuations it was common to value each year's business separately, even where it involved fifty or sixty schedules of whole-life business distributed over a very wide range, it was evident that in some circumstances select valuation was more convenient than aggregate valuation. He was inclined to think that even in ordinary valuations the gain would more than offset any inconvenience from the larger number of groups and from the increase in the number of multiplications, &c., required at valuation. Mr. Lidstone had shown that where endowment assurances were grouped by years of entry and original terms it was possible to obtain the valuation ages from the pure premiums and thus save the tabulation of the Z's. This would reduce the work of classification, although the tabulation of valuation factors dependent on entry

ages could not be avoided entirely unless, as suggested by Mr. Austin in a recent discussion, the mean entry ages could be satisfactorily obtained from the office premiums. A further advantage of classification by years of issue arose in certain cases where the bonuses allotted were the same for all policies of the same duration and sum assured. He believed that the general practice was to maintain, either on the valuation cards or on some supplementary cards or registers, a record of the bonuses allotted to each policy. With annual distributions of profit the maintenance of that record became a serious burden, and it had been found practicable by sub-dividing the usual valuation groups according to the years of issue of the policy to dispense altogether with any record of the bonuses on individual policies except where variation was introduced by the surrender of bonus or its application to premium reduction.

Mr. A. LEVINE said that select valuations were no doubt right in theory, but they were not very practicable, and even if they were he doubted whether they would be correct under the conditions of modern life assurance practice. He particularly referred to select valuations of endowment assurances. He believed it was not unusual now-a-days to take at ordinary rates proposals for endowment assurance policies which would not be accepted at ordinary rates under a whole-life table. For instance, if a man was a little above the average weight and made a proposal for an endowment at 50, he would be taken at tabular rates, whereas under a whole-life table he would be rated up. He thought the effect of practice of that kind was that the endowment assurance class now was not absolutely a select class. Then again, there was a very growing practice of accepting special classes of persons without medical examination. It would not be possible without far more trouble than it was worth to keep all those lives outside the ordinary endowment assurance groups, and if they were included it would not be right to value them by an $E^{(M)}$ select mortality table. For those reasons he thought actuaries would continue to make their valuations by the ordinary aggregate table. Moreover, the author's tables showed that the results which would be obtained by making a select valuation did not differ very much from those obtained by making an aggregate valuation.

Mr. R. E. UNDERWOOD had hoped that the various proposed methods of approximate valuation would make select valuation a practical possibility, but when the author pointed out that the mortality varied not only with the duration of the policy but with the original term of the assurance, he felt that they were thrown back upon the aggregate table again. Moreover, in view of the great change that had taken place in mortality, the Select Table could not be regarded as accurate and they might almost as well value by the aggregate table as heretofore. Even when they had more up-to-date tables on which greater reliance could be placed, it might be that in practice it would not be found desirable to value by select tables. The question of expense might lead to an aggregate valuation being far more convenient. At the same time it would be useful to the

actuary, while publishing his valuation report on an aggregate basis, to be able to make a select valuation by approximate methods, so that he might know what the position was even if he did not care to put it forward publicly. Mr. Levine had referred to the fact that it was sometimes the custom to accept policies at ordinary rates for endowment assurances which would not be taken for whole-life assurances. If he understood the position rightly, the doctor in recommending the actuary to accept a policy under an endowment assurance table at ordinary rates did so because he considered that, if there was any extra risk, that risk was outside the term of the endowment assurance. Therefore, it appeared that such lives might be considered as ordinary select lives for endowment assurance, although they might not be for whole-life policies.

Mr. G. W. RICHMOND thought that in practice it was necessary to compromise between two objects—one to conduct a practical valuation and the other to have at disposal the results of an investigation into the effect of the many variable factors that underlay insurance business. He would, therefore, welcome the most minute investigations into those factors, not with the view necessarily of applying them direct in ordinary office valuations but as a matter of study by which it was possible to learn how to avoid constant errors. A good example of what he had in mind was the investigation which showed that the combined H^{M} and $H^{M(5)}$ valuation afforded a very satisfactory approximation to a select valuation. But what was called selection had mixed up with it a number of other factors, and it was not possible quite to distinguish the relative weight of each and how far selection was a continuing feature and would be reflected in the future. The author had not referred to a paper that had appeared in one of the Congress *Transactions*, in which the Swedish actuary, Mr. Stolz, brought forward some very elaborate graduations of the British Offices' endowment assurance experience. He might perhaps have availed himself of the benefit to be derived from Mr. Stolz's work.

Mr. H. J. P. OAKLEY doubted whether the preparation of tables based on $E^{(M)}$ experience was necessary for the author's purpose. He could have followed out his argument by means of the $O^{[M]}$ Select Tables. The $E^{(M)}$ Select and the $O^{(M)}$ Select were not truly contemporary, although they covered the same period and were furnished by the same offices, for whereas the $O^{(M)}$ Select was spread fairly evenly over the period from 1863 to 1893, the $E^{(M)}$ was concentrated at the end of the period. If the mortality was improving during the period 1863 to 1893, then the $E^{[M]}$ Select dealt with the most favourable portion of the period. Taking the figures of one large office, he found that in the middle of the '70's they had only 700 endowment assurances out of a total of 14,000, and fifteen years later they had something like 6,000 out of 32,000 contracts. Again, looking at the ungraduated data of the $E^{(M)}$ Select experience, it would be seen how heavy were the numbers going out as existing in the early years, so that the experience was really comparatively recent. Not only so, but when the five years select

period was excluded the balance remaining was small and not very far from being select. Then again the $O^{[M]}$ Select was based on with-profits and the $E^{(M)}$ on with-profits and non-profits. The author said that the low rates under the $E^{(M)}$ Select would be found to persist, which opinion he gave in opposition to those who held that the low rates could not be maintained now that that class of business had become the vogue. His own experience confirmed the result of the author's investigations ; but he thought it would be found to persist largely by reason of the general improvement in mortality. He thought it might be expressed in the following way, that whereas from 1863 to 1893 the whole-life with-profit policy was the normal contract and there was selection of endowment assurance by a small minority more likely to survive, in the ensuing thirty years the endowment assurance had become the normal contract and there was selection of whole-life policies by a minority who were less likely to survive, although they might still show an improvement on the previous period. In any case, the argument for the use of select tables need not have been confined to endowment assurances.

Turning to the Model Office, it was not quite clear what office premiums the author had used or whence his funds had accumulated. They had been built up by adding to the reserves a definitely calculated surplus on the basis of a 30*s.* simple reversionary bonus. He thought that the author would have attained his end quite as well had he dealt with reserves—which were the important things. In recent years there had been a swing of the pendulum, from with-profit to non-profit, and the with-profit policyholders were very interested in the contracts of the non-profit policyholders, and the reserves under the non-profit policies gave really as much concern as the reserves under the with-profit policies. Indeed, in recent years it had been the loading on the with-profit policies that had helped to maintain reserves. There had been no bonuses, and when in course of time, as was hoped, there would again be profits from non-profit business, they would assist the bonuses.

Notwithstanding the difficulty of setting up reserves at a time when a large new business imposed a heavy strain, the author suggested still larger reserves because of the select mortality experienced. He advised giving up the aggregate for the select, but he took no account of initial expenses. The aggregate valuation released funds which would have had to be reserved according to select mortality and eased the strain of initial expenses. The valuation basis, as shortly expressed by an aggregate mortality table at a certain rate of interest was not the whole story. It provided a useful standard and made for expedition and simplicity, but various adjustments still fell to be made—some of them empirical—and the paper enabled the formation of conclusions so that behind the published standard the actuary could have a knowledge of the facts and shape his course accordingly.

The PRESIDENT, in proposing a vote of thanks to the author for his very interesting paper, said that it appeared to him to be a very suggestive and instructive contribution. He felt there was

much to be said on the lines indicated by Mr. Oakley. It was the old question of the dependence of the value of conclusions upon the bases on which the conclusions were formed. Every thoughtful student of the subject would agree with Mr. Oakley that great importance attached to the fact that the endowment assurance mortality statistics related to a more recent period on the average than the bulk of the statistics relative to the mortality among whole-life assurance. It was very generally accepted that throughout a long period the rate of mortality had been improving, and he thought it was very fairly to be argued that much of the difference between the endowment and whole-life assurance tables of mortality was due rather to the period round which the weight of the observations was centred in each case than to some inherent difference in the experience.

Mr. F. W. FULFORD, replying on behalf of Mr. Brown, believed that there was still a considerable difference, due to selection, between whole-life and endowment assurance mortality. Instead of regarding (as had been done in the past) a proposer for endowment assurance as exercising self-selection, he would regard a proposer for whole-life assurance as exercising an option against the office. It might be found in the next investigation that the mortality under long-term endowment assurances was closely allied to that under whole-life assurances and the two classes might even be combined. Such points could not be decided beforehand, but it would certainly seem worth while in the next investigation to divide the endowment assurances according to term for preliminary investigation.

An aggregate net premium valuation could not now be justified except on grounds of convenience, and when they employed an arbitrary method of valuation it was important to make an investigation, such as the author had done, to appreciate the financial effect. The paper, perhaps, was also useful in pointing out the importance of considering the effect on future profits if the valuation basis was to be changed. When an office had been valuing on a certain basis for many years an actuary knew how the profits emerged from time to time, and before a change was made in the valuation basis he must satisfy himself that his profits would emerge in future, because a change of valuation basis did not simply mean putting up the necessary extra reserves. If the mortality rate on the new basis increased in a different ratio from the old, the reserves would be built up at a different rate and the profits would be realized differently. Mr. Oakley had suggested that the fund might have been built up on another basis. The author had hoped to build up the reserve fund by means of a bonus reserve valuation, but had not found time to do so, owing to ill-health and pressure of business. The net effect would probably have been to accentuate the differences, *i.e.*, the bonuses would have been too large in the earlier years with a corresponding decrease later. If higher bonuses were paid at the outset they must be smaller in future.

Local Government and other Officers' Superannuation Act, 1922.

[CONTRIBUTED.]

THIS Act, which received the Royal Assent in August last, is of considerable importance to actuaries, and it is thought that a short precis of its main provisions together with some account of the steps which led to its introduction may be of interest to the readers of the *Journal.*

Prior to the passing of this Act there was no general authority for the superannuation of officers of local authorities. Enactments have from time to time been passed conferring superannuation rights upon specified classes of employees, for example, teachers, officers of poor law authorities and police, but, except in London, where the Superannuation (Metropolis) Act, 1866 gave a discretionary power to award superannuation allowances in certain cases, no local authority could establish a superannuation scheme unless it first obtained special powers under a Local Act of Parliament.

For some years prior to the war local government officials had been pressing for a comprehensive superannuation scheme and a strong departmental committee was set up at the end of 1918 to consider whether it was desirable to introduce a scheme of superannuation applicable to the persons in the employment of local authorities in England and Wales ; on this Committee the profession was represented by Sir Alfred Watson and Mr. W. Palin Elderton. The Committee reported in July 1919 (Parl. Paper—Cmd. 329) in favour of a compulsory and contributory scheme embracing practically all established officers and servants. The benefits recommended were pensions of one-sixtieth of the average salary during the last five years of service for each year of service (with a maximum of 40/60ths) to commence at age 65, or on the completion of forty years of service, but not earlier than age 60. The basic rate of contribution, divisible equally between the officers and the employing authorities, was fixed at 10 per-cent for entrants of all ages, and a return of the officer's contributions was to be made on his death, and in certain cases on withdrawal. The Committee recommended that the contributions should be funded, but not in a single fund for the whole country ; they expressed the view that a fund should be set up in each of the larger towns and that the other local authorities in each county

should form a joint fund. They further recommended that any deficiency which might from time to time emerge in any of the funds on valuation should be made good by the authority concerned, without calling upon existing officers to make any additional contribution, though in the case of officers subsequently appointed it was suggested that the appropriate rate recommended by an actuary as a result of the valuation should be fixed on the half and half basis. Somewhat similar provisions were proposed in the case of a surplus. A novel feature of the scheme recommended by the Committee was that where an officer moved from one local authority to another a transfer value should be carried from the old fund to the new in order that the authority to which he transferred might give him credit for his back-service without incurring loss. With regard to back-service before the commencement of the scheme the Committee were of opinion that authorities should be empowered to give back-service rights (based generally on 1/120ths) out of the current rates, or out of the superannuation fund if appropriate arrangements in the form of a deficiency contribution were made.

Prior to the war it is understood that local government officers generally were in favour of a contributory scheme of the kind recommended, but for various reasons, probably not unconnected with the grant of non-contributory pensions to teachers in 1918 (recently changed to a contributory system by the Act of 1922), the scheme did not receive the universal support which had seemed likely, and no general legislation to give effect to the recommendations of the Committee was introduced. It is interesting to observe, however, clear indications of the Committee's work in the recent Bills promoted by local authorities wishing to set up superannuation schemes for their own officers ; in many of these the rate of contribution was fixed at 10 per-cent as recommended by the Committee, this figure being considerably higher than that customary hitherto for benefits of the type in question.

The Superannuation Bill of 1922 was introduced by a private member, Sir Herbert Nield, who was fortunate enough to secure a high place in the Members' ballot ; otherwise the prospects of the Bill reaching the statute book would have been very small indeed. Considerable changes were made as the Bill passed through Committee and in the main the provisions of the Act follow the recommendations of the Departmental Committee. The benefits and contributions are almost identical, but in other

respects there are important differences. The most important of these is that the Act is *permissive* and though any local authority may adopt it, the scheme is not universal and any authority may stand aside. It followed, therefore, that the proposals of the Committee designed to bring all local authorities into a relatively small number (about 200) of funds could not be followed, but each authority having a minimum number of officers or servants (fixed at 50 in the Act) may adopt the Act and set up its own fund. Before so deciding, however, the local body must be fully informed as to the burden to be incurred, and the Act lays down definitely that an actuarial estimate, certified by a qualified actuary, being a Fellow of the Institute or of the Faculty, must in each case be obtained before the authority passes its adopting resolution. This is particularly important having regard to the provision elsewhere in the Act conferring back-service rights on all persons in the employment of the authority at the date of commencement, and laying down that the initial deficiency thrown upon the fund is to be made good by an equalised annual payment out of the rates over a term not exceeding 40 years.

In regard to back-service it may be mentioned that while the Act fixes a general scale of 1/120th for previous service, any authority may in any particular case by resolution give a higher scale (not exceeding 1/60th) but in this event the excess has to be charged directly on the rates as an emergency cost and not on the fund, which is thus protected from the loss which might fall on it, were a liberal policy adopted after the initial deficiency and the equalized charge to redeem it had been fixed.

There remains the case of the smaller authorities having less than 50 officers and too small to maintain a stable fund of their own. Here the Act permits combination but only within the county boundary; the council of any county may combine with any other local authority in the county, or two or more local authorities in a county may form a joint fund. In such a case a scheme of combination has to be prepared to secure equity between the several partners, and having regard to the strain which might be thrown upon the joint fund were one of the constituent authorities to adopt a less stringent standard of administration, *e.g.*, with regard to the conditions of retirement in ill-health, than that of its associates the importance of this provision (Section 5 of the Act) is evident. Similarly the apportionment of a deficiency on valuation in such a case will

present problems in which the advice of the actuary will be needed. Difficult cases may arise, for example, where there is a dispute as to whether the deficiency has arisen in whole or in part owing to the laxity of the administrative standards of one or more of the partners.

Another method of dealing with small authorities is possible under the Act (Section 5 (3)), namely, by contract with a larger authority; this will doubtless be used where numbers are very small and the machinery of combination and joint control would not be suitable. In such a case the officers of the smaller authority will in effect be treated as officers of the larger authority which may, however, impose such terms and conditions as it thinks fit.

In future the minimum period of qualifying service for superannuation will be 10 years of contribution; but in the case of present officers back-service will be included, and where an existing officer is retired on grounds of ill-health before he has contributed for 10 years the whole of his service will rank for pension as non-contributory service. The similar case will not arise in respect of age retirements, as existing officers aged 55 or over will not be called upon to contribute, the whole of their service whether before or after the appointed day ranking as non-contributory service. With regard to pensions to be granted in the first few years of a scheme—and there will probably be many of these as, except in very special circumstances, retirement will be compulsory at age 65 even for existing officers—mention may be made of the requirement of the Act that where salaries have not been stabilised and a system of fluctuating bonus obtains, that part of the pension arising from the latter is to be subject to variation in accordance with the rules in force for Civil Servants' pensions.

With regard to officers moving from one authority to another the Act follows the recommendations of the Departmental Committee, and transfer values will be passed between the funds concerned.

In the case of authorities which already have a superannuation scheme in being the Act may be adopted, but in this case the rights of existing officers have to be safeguarded by a scheme under which, *inter alia*, the transfer of the existing assets will be arranged.

Attention may be drawn to an interesting provision of the Act, securing to each authority adopting it the right to decide

which posts they desire to designate as pensionable. An authority may thus include all its regular staff or only certain sections of it. Further posts may be designated subsequently if the authority desires, but in this case it must, generally speaking, conform afresh with the initial procedure of adoption and obtain an actuarial estimate of the additional cost. Permission to have separate funds for officers and servants respectively is conferred by the Act in certain cases; and it is not unlikely that some of the larger authorities may wish to take advantage of this provision.

The Act lays down that each fund is to be valued at quinquennial intervals by an actuary and follows the recommendations of the Departmental Committee with regard to the liquidation of deficiencies and revision of rates of contribution for future entrants.

LEGAL NOTES.

By Robert Allen Bateman, B.Sc. (Econ.), *Barrister-at-Law.*

Mathieson v Atlas Assurance Company, Ltd. & Others (not reported).

Under what circumstances will a competent Court order a decree for foreclosure absolute to be re-opened?

As this case will probably not be reported in any of the ordinary series of law reports, and as decisions were given both in the Court of Appeal and in the House of Lords, it will be most useful to give the actual judgments in all the Courts, so that these may serve as an authentic reference for any future cases of a similar nature. The facts are sufficiently set out in the judgment of Peterson, J., before whom the Motion to reopen came originally in the Chancery Division. He said: "This is a "Motion to reopen a foreclosure decree absolute which was made "as far back as the 19 April 1916. The applicant, the mortgagor, "has a leasehold interest in what is known as Cross Keys "House, and three houses in Moorgate Street, Nos. 56, 58 and "60. As to the houses in Moorgate Street they were held on "leases, or underleases, which expired in 1916 with an option "exercisable within, I think it was, two years after 1916 or the "termination of the underleases—at any rate, it was exercisable "about that period—under which the mortgagor was at liberty to "undertake to rebuild the houses, and on rebuilding the houses

"was entitled to a lease for a considerable term in accordance "with the provisions of the building agreement. So far as that "part of the interest was concerned there was, therefore, a "considerable amount of speculation in it. It was really a "comparatively short term with the option to enter into a "building agreement at the end of it. As to Cross Keys House "there was a long term.

"These properties were subject to three mortgages. The "first was a mortgage to the Atlas Company; the second was a "mortgage for some £8,000 in favour of second mortgagees; and "there was a third mortgage in favour of the mortgagor's "brothers. One of the premiums under an endowment policy "which was part of the security having failed to be paid by the "mortgagor after the war had begun a Receiver was appointed "by the Atlas Company and they took proceedings for foreclosure, "and in April 1915 the usual order nisi was made in their action. "In July 1915 the certificate was made finding that a sum of "£31,000 odd was due to the Atlas Company. In April 1916 there "was the usual application under the Courts (Emergency Powers) "Act for leave to proceed to obtain the order absolute, and on "that application Mr. Justice Eve directed that if within a "limited time the mortgagor deposited £500 to secure any costs "that might be thrown away, he should have some further "period for the purpose of endeavouring to sell the property and "realize what he contended was the then value of the property. "He however failed to deposit the £500, and on the 19 April 1916 "the order absolute for foreclosure was made. It is to be "observed that in those proceedings the second mortgagees and "the third mortgagees were parties, and at that time neither of "the mortgagees considered, or apparently considered, that it "was worth their while to redeem the first mortgagees.

"The order for foreclosure having been made on the 19 April "1916, and the foreclosure proceedings having been taken "largely, I am told, by reason of the fact that it had "become necessary to determine whether the option to enter "into the building agreement should be exercised or not, in "August 1916 the Atlas Company actually entered into a "contract with the lessors by which they undertook the "obligation to rebuild the premises in Moorgate Street. Nothing "happened until the end of 1919. There appear to have been "various endeavours to sell on the part of the Atlas Company. "One of the offers they received apparently was £22,500.

"Another was an offer of over £30,000, which, however, was "withdrawn when it was discovered that some of the information "which had been supplied was supposed not to be accurate. "At the end of 1919, however, the Atlas Company entered into "a contract for sale with Mr. Manning's clients (*i.e.*, the London "& Northern Estates Co., Ld.), the price being £40,000. That "was in December of 1919. Under that contract there was a "provision that the Atlas Company should be indemnified "against the liabilities it had undertaken in connection with the "rebuilding of the premises in Moorgate Street.

"It was not until the 18 June 1920 that Mr. Mathieson, the "mortgagor, took any steps at all. Then by his solicitors he "wrote to the General Manager of the Atlas Company a letter "which said: 'our client would like to know whether if he pays "'your Company the amount due for principal, interest and "'costs your Company would be inclined to re-open the matter?' "The answer to that was that the property is in fact now the "subject of a contract for sale, and quite shortly afterwards the "present Notice of Motion was launched. That means that over "four years after the order for foreclosure absolute had been "made, and six months after a contract for sale by the Atlas "Company had been entered into, the mortgagor has taken "proceedings for the purpose of re-opening the foreclosure.

"I have had a number of authorities called to my attention, "some of them very early ones and reported so briefly or in such "a way as not to be of very much assistance to the Court. There "are, however, one or two authorities which throw some light— "one of them very considerable light—on the way in which the "discretion of the Court ought to be exercised in a case of this "description. There is no doubt that, as Sir George Jessel "pointed out in *Campbell* v. *Holyland*, and as has been pointed "out in other cases, there is jurisdiction in the Court in a proper "case to re-open foreclosure even after the order absolute has "been made. But there are various cases, and in the first to "which I will refer, *Thornhill* v. *Manning*, there are illustrations "given of the way in which that jurisdiction ought to be "exercised. In *Thornhill* v. *Manning* the Lord Chancellor said "this: 'It is quite impossible to lay down any general rule as "'to the circumstances which will induce the Court to re-open "'a decree of foreclosure, but this I must observe that the "'Court has a very strong inclination to give assistance to a "'mortgagor if he applies promptly.' The observations the

"Lord Chancellor is there making obviously have relation to the "simple case where the question is whether the foreclosure ought "to be re-opened as against the mortgagee, the question not being "complicated by the fact that the mortgagee has in fact contracted "to sell to a purchaser. In *Patch* v. *Ward*, Sir John Rolt said "this—it is, I think, in the nature of *obiter dictum* and is "perhaps a little bit too widely expressed having regard to the "other decisions—'I asked counsel for the appellant several "'times for authority to show upon what principle the Court "'would open foreclosure', &c. (reading to the words) "'exigencies of the order.' I think that is perhaps expressed in "such a way as to disregard some of the cases where relief has "in fact been given, but at any rate it points to this, that in "the opinion of Sir John Rolt, Lord Justice, it was not the right "of the mortgagor simply to come to the Court and say 'I am now "'ready to pay, and I want to have this foreclosure re-opened.'

"Then the next case I come to is, I think, the most important "of all of them, this is *Campbell* v. *Holyland*, the case which was "before Sir George Jessel. It is to be observed that in that "case according to the statement of facts, the person who is "referred to as the purchaser was merely the purchaser of the "mortgagee's interest, that is to say, the mortgage debt. He "was not a purchaser from the mortgagee under his power of "sale, and it was a purchase which was effected before the order "for foreclosure absolute had been made. In the particular "circumstances of the case the Master of the Rolls opened the "foreclosure and allowed a further time for redemption. The "actual facts are not very material for the purposes of the "present case. The Master of the Rolls, however, does make "some observations which I think are of considerable assistance "and guidance in the present case. First of all the Master of "the Rolls, after giving a very valuable explanation of the "historical grounds on which the order for foreclosure can be "opened, goes on to say that under what circumstances the "discretion of the Court should be exercised is quite another "matter, and he points out that the mortgagee had a right to "deal with an estate acquired under foreclosure absolute the "day after he acquired it, but he, of course, knew there might "be circumstances entitling a mortgagor to redeem, and the "person who bought from a mortgagee must be considered to "know that in proper circumstances foreclosure might be "re-opened. Then he goes on to enquire what are the terms on

" which the judicial discretion ought to be exercised. Then he " points out, as was pointed out before, I think, in *Thornhill* v. " *Manning*, that that question must depend upon the circumstances " of each case, and he also insists upon the fact that in the first " place the mortgagor must come as it is said 'promptly', that " is, within a reasonable time, and he goes on to say: 'He is " 'not to let the mortgagee deal with the estate as his own' " &c. (reading to the words) 'he must come within a reasonable " 'time.' Then he goes into the question of what is a reasonable " time. He points out that for the purpose of answering that " question you must first have regard to the nature of the " property, for instance, whether it is an estate in possession, or " an estate in reversion; secondly, whether the mortgagor was " by some accident prevented from redeeming. One of the " instances he refers to is obviously one of the cases—where the " mortgagor had provided the money but his messenger was " unable to find any place on the coach which was to take him " up to London, so that the money was not in fact paid on the " appointed day. Then he says an element of consideration has " always been the nature of the property as regards value, and " he gives two instances, one a ground on which foreclosure " might be re-opened, and the other a ground on which there " might be considerable difficulty in re-opening it. For instance, " he says if an estate worth £50,000 is foreclosed for a mortgage " debt of £5,000 and he contrasts that with a foreclosure in " respect of an estate worth considerably less than the mortgage " debt; and he says there may be not only money but other " considerations, and he calls attention to a family estate, chattel, " or a picture having a special value to the mortgagor. In the " first place he holds that it is absolutely necessary for the " mortgagor if he desires to re-open to come promptly. The " second point is this. It is said by the Master of the Rolls that " you must not interfere against purchasers. Then he says this " in a passage which I will read: 'As I have already explained, " 'there are purchasers and purchasers' &c. (reading to the " words) 'I am of opinion that it would not.' Then towards the " end he says that in the particular circumstances of the case he " has before him he is of opinion that the mortgagor has been " sufficiently prompt, and then he goes on: 'As I said before, I " 'by no means say', &c. (reading down to the words) 'weigh " 'with the Court in opening foreclosure.'

" Bearing in mind those observations I have applied those

"observations to the present case, having regard also to the "expression of opinion by Mr. Justice Chitty in *Abdy* v. *Brown*, "which is only reported in the Solicitor's Journal for 1894. "That was not the case of a sale by the mortgagee; it was an "application by the mortgagor to re-open foreclosure when the "order for foreclosure absolute had been made in August 1891, "that is to say three years afterwards. It was a mortgage of a "share in a reversionary property, and the foreclosure went for "£5,600. In June 1892, that is to say, rather less than a year "after the order for foreclosure absolute, the tenant for life died, "and the mortgagees received a sum which left them after all "deductions £7,400. It did not appear that the security "exceeded the value of the debt at the time of the foreclosure, "and there was a special circumstance that the mortgagor had "received £500 from the mortgagees in response to appeals to "them to make him a donation, and acknowledged it to be a "gratuity, and Mr. Justice Chitty says that the exercise of the "jurisdiction by the Court was a matter of judicial discretion, "and the note of his judgment goes on to say that his Lordship "knew of no case where foreclosure had been open after so long "—that is three years. (Reading a passage). Then he deals "with the special facts of the case, that is to say, the appeal for "the gratuity of £500, and says that he thought it would be a "breach of good faith if he acceded to the mortgagor's "application, and on all grounds referred to he declined to "exercise the judicial discretion. What have I in this case? "There is no suggestion of unfair or improper conduct on the "part of the mortgagee. If there were any such suggestion I "think that there was absolutely no ground for it at all. I refer, "of course, to any suggestion of there being unfair or improper "conduct by the mortgagees in any of their actions in connection "with the obtaining of the order for foreclosure absolute. The "real grounds of the claim in the present case are two, first, that "since the order for foreclosure absolute, or rather since the date "of the Armistice, there has been a rise in the value of the "property in question; and secondly, that the financial position "of the mortgagor has now improved to such an extent that he "is at present in a position to redeem. As against that, I have "this that no steps were taken at all to re-open the foreclosure "for over four years after the foreclosure absolute; that the "property was *bona fide* sold in December 1919, to a purchaser "for the best price that the vendors, the mortgagees, who had

"obtained the order for foreclosure absolute, could then obtain—"£40,000. I have also this, that the mortgagees ever since the "date for foreclosure absolute in April 1916, have acted as the "owners of the estate. They incurred liabilities as owners of "the property ; they entered into a contract about August "1916, under which they bound themselves to rebuild the "premises in Moorgate Street. In those circumstances would it "be right on the part of the Court to exercise the discretion in "favour of the mortgagor, and to re-open the foreclosure on the "grounds I have suggested, having regard to the fact that no "steps have been taken by the mortgagor for the space of over "four years ; that the mortgagees have acted and incurred "liabilities as owners of the estate on the faith of the order for "foreclosure absolute, and that the interest of a third party has "intervened, namely, that of a purchaser who has acquired an "interest under a contract for sale in the property, having "agreed to purchase it for a sum of £40,000 without any notice "or suggestion that the mortgagor intended to intervene and "endeavour to re-open the foreclosure that had taken place "nearly four years before. The circumstances of this case are "such that I think it would be the worst possible example if I "were to re-open the foreclosure in such a case. The result "would be that it would be almost impossible for a mortgagee "who had obtained an order for foreclosure absolute ever to "made a title to the property which he had acquired by virtue "of the order for foreclosure, or at any rate to make a title "within—if I were to decide this case in favour of the mortgagor "—4 or 5 years after the order for foreclosure absolute. In my "opinion there is no case at all which binds me or suggests that I "am bound to give the relief asked for in this case. On the "contrary, I think that the observations of Sir George Jessel "in *Campbell* v. *Holyland* point clearly to the fact that if this "case had come before him he would not have considered the "question of granting relief to the mortgagor for one instant.

"The result, therefore, is that in my opinion the application "in this case fails and must be dismissed."

The Mortgagor appealed and the appeal was heard by the Master of the Rolls (Lord Sterndale), Lord Justice Warrington and Lord Justice Younger. In giving his reasons for dismissing the appeal the Master of the Rolls said : "This is an appeal from "a decision of Mr. Justice Peterson on a motion to re-open a "foreclosure decree absolute which was made on the 19 April

"1916. The property in respect of which the application is "made is property known as the Cross Keys House and also "three houses in Moorgate Street. The houses in Moorgate "Street were held on leases which expired in 1916 with an option "which had to be exercised, I think, before July 1916, although "the learned judge I think states inaccurately within two years, "under which option the mortgagee was at liberty to undertake "to rebuild the houses, and if he did so he was entitled to a "lease for a considerable time—I think the same time as the "lease under which he held the Cross Keys House. The Cross "Keys House was held under a lease for a considerable term. "There were three mortgages on the property; the first to the "Atlas Company; the second to some mortgagees whose names "I do not think have been mentioned; and a third in favour of "the mortgagor's brothers. A receiver was applied for and "obtained by the Atlas Company, not by reason of the default "in payment of any instalments of interest, but by reason of a "default in payment of a premium under an endowment policy "which composed part of the security. They then, after "obtaining the appointment of a receiver, took proceedings for "foreclosure and an Order nisi was made in April 1915, and "shortly afterwards a certificate was made finding a sum of "£31,000 due to the Atlas Company. In April 1916, an "application was made to Mr. Justice Eve under the Courts "(Emergency Powers) Act for leave to proceed, and Mr. Justice "Eve directed that if the mortgagor were to deposit £500 to "secure costs thrown away within a certain period he should "have time given him to endeavour to sell the property and "realize what he contended was its value. He did not do so "and the consequence was that on the 19 April 1916 an Order "absolute for foreclosure was made.

"I quite agree with what has been argued by both learned "Counsel for the appellant that it is not necessary in order to "enable us to re-open this foreclosure to say that that Order of "Mr. Justice Eve was wrong. If it were necessary to say so, "it might present a great difficulty because that Order was never "appealed against, but I agree, I do not think it does conclude "the matter in any way whatever. At the same time I also "think it is a matter which cannot be put out of consideration "altogether. It may be, and I think it very likely is the case, "that Mr. Justice Eve thought it right to allow the Order "absolute to be made because it was necessary to decide whether

"the option to which I have referred should be exercised or not. "If it were not, the property might be—probably would be— "considerably depreciated in value. The Atlas Company, the "first mortgagees, did exercise that option to rebuild and they "entered into an agreement with the lessors in August of that "same year by which they put themselves under an obligation "to rebuild those premises. I think they obtained an extension "of time for doing it, but they put themselves under an "obligation to rebuild and they tried to sell the property. They "could not sell the property for a long time. But towards the "end of 1919, that is to say, more than three years after the "Order for foreclosure absolute, they did enter into a contract "with the London & Northern Estates Company at a price of "£40,000 and by that contract the purchasers were to indemnify "the Atlas Company against the liabilities undertaken by their "obligation to rebuild. Nothing was done by the mortgagor "at all until 1920 when he wrote to the Atlas Company asking "if they would allow him to pay off what was due to them and "to re-open the matter. They said to him that they had sold "the property and they could not do it and thereupon these "proceedings were started, and the strong ground upon which "the learned Counsel for the appellant put the matter was this, "that the state of impecuniosity to which the mortgagor was "reduced at the time of the commencement of the foreclosure "proceedings and the making of the Order, was brought about "by the world-wide catastrophe of the war and that that being "so, he ought now, when matters have become somewhat "different or very different with regard to his financial position, "to be allowed to re-open the matter. That his position was "made worse by the war I have no doubt, but that the position "was so enormously altered, as was rather suggested in the "first instance, I cannot see, because on taking the appellant's "own affidavit, I see that he says that at the outbreak of war "he enjoyed a considerable income from shares in a certain "printing firm, that he held 11,154 preference shares, 10,000 "ordinary shares and 90 deferred shares, but he also goes on to "say that of these he had transferred 11,154 preference, 10,000 "ordinary and 70 deferred to certain persons as a security for "money advanced to him, and that these persons were entitled "under this transfer to receive all the income from the shares "except that certain payments were to be made to himself and "his wife and family, amounting to about £1,100 a year, and

"therefore his position was not quite that of the great affluence "that was suggested at the opening of the case, but still he was "getting a substantial amount from these shares mortgaged as "they were and he says, and I should think quite rightly, that "the result of the war was to produce a very great depression in "the trade and that the income of shares dropped from about "£4,000 to about £450 a year. Therefore, his position "undoubtedly was made very much worse by the war. He also "says, and I have no doubt rightly, that since the Armistice, or "since the war was over, if it is over, trade had revived and he "is now again receiving a substantial amount from these shares "and that he has paid off, I think he says, the mortgagees to "whom the shares were assigned.

"But the real circumstance that has made the difference "here is this. As everybody knows, there has been a very large "appreciation in the value of property such as this in recent "months. Probably the peak has been reached and the progress "is now rather downwards, but at any rate there has been a very "large appreciation in the value of property and that probably "made it possible for him so to finance the transaction that he "might find the money in order to pay off the Atlas Company. "Now the question is whether he ought to be allowed to do so: "Mr. Justice Peterson has held he ought not. I have no "intention of discussing all the cases that were cited to us. I "do not think it is in the least necessary because I am quite "satisfied that we have ample jurisdiction to entertain this "motion if we think it proper to do so. It is quite true that it is "four years since the Order absolute. Lapse of time is not an "absolute bar to a motion of this kind. Cases were cited to us "in support of that, which were no doubt under somewhat "different circumstances. There was one case, *Cocker* v. *Bevis*, "1 Cases in Chancery 61; 2 Freeman, 129; 22 English Reports, "695, where there was an interval of 16 years, but it is true that "during those 16 years the Commonwealth had intervened and "the Monarchy had disappeared and at the time the motion "was brought the Monarchy had been restored and the plaintiff "was alive. Those are circumstances which no doubt do not "exist in this case, but still I am quite satisfied that four years "lapse of time is not an absolute bar at all. Then, again, I am "quite satisfied that the fact that there has been a sale by the "mortgagee and that the interests of the purchaser have to be "regarded is not of itself either a bar. We have perfect

"jurisdiction to do it. But the serious and important question "is, ought we? In my opinion, we ought not. It seems to "me that the time is a matter to be considered. It seems to "me also, although perhaps it is not of very great weight, that "the fact that leave was given to make the Order absolute under "the Courts (Emergency Powers) Act is also a matter to be "considered. No doubt the learned Judge had not the "circumstances—they did not exist—which we have before us "and no doubt it is not quite in *pari materia*, but still we have "the fact for what it is worth that the mortgagor, having the "opportunity of putting before the Judge reasons why this "Order should not be made, the Judge made it, and we have "also, for what it is worth, although perhaps the weight is not "very great, this, that as has been pointed out by Sir George "Jessel (*Campbell* v. *Holyland*, 7 Ch. Div. 166), a purchaser "from a mortgagee after foreclosure absolute knows that that "foreclosure may be re-opened and he takes his bargain with "that before his eyes; but it is something if he knows that there "is a procedure provided to protect the mortgagor from that "Order being made, and under that procedure leave has been "given to make it. These are circumstances to be considered.

"But there is another circumstance which seems to me "extremely important and that is this. First, this sale was "made long after the Order for foreclosure absolute was made, "over three years, and, secondly, before that had been done, "the Atlas Company had by reason of the exercise of their "option brought themselves under very serious obligations to "the ground landlords of this property. By their agreement of "sale they have got an indemnity with which no doubt they "are satisfied from the purchasers. Suppose this matter be "re-opened, they will remain liable to those obligations, but if "it be entirely re-opened, they lose that indemnity and they have "left to them only the indemnity of the mortgagor which may "be very good or which may not be very good; at any rate, "one which they ought not to have forced upon them against "their will.

"For these reasons, which are practically identical with "those mentioned by Mr. Justice Peterson, I think that this "Order ought not to be made and I do not think either, that the "alternative Order suggested by the appellant, namely, an "Order to re-open but subject to confirmation of the sale, ought "to be made. I do not think this is a case in which the mortgagor

"has shown a proper case for re-opening the foreclosure at all. "And besides, although no doubt he will benefit if such an Order "as I have mentioned were made, the benefit is not anything "equal to what he is asking for by the other Order, and I do not "think he has made any case to justify the alternative Order "either.

"For these reasons I think the appeal must be dismissed."

From this decision there was an appeal to the House of Lords which was heard by Lords Buckmaster, Atkinson, Sumner, Wrenbury and Carson. In delivering the judgment of the House, Lord Buckmaster said :

"My Lords, that the discretionary power to re-open a decree "for foreclosure of mortgaged property exists is beyond dispute, "and no conditions need to be established as essential or "precedent to the exercise of that power, but the discretion is "judicial and requires to be sparingly and carefully exercised. "The circumstances to which regard must be had for the purpose "are well-known and were clearly stated by Sir George Jessel in "the case of *Campbell* v. *Holyland*, 7 Ch. Div. at p. 166. But "when such circumstances have been properly apprehended by "a Judge who has accurately investigated the facts and applied "his discretion to the circumstances so elucidated, neither the "Court of Appeal nor this House will lightly interfere with what "he has done.

"In this Appeal no case has been established to show that "any of the circumstances were interpreted too unfavourably "to the appellant by Mr. Justice Peterson or the Court of Appeal, "or that the application was dealt with on a wrong principle. "Further, I may add that a careful investigation of the facts, in "which this House has been materially assisted by Counsel for "the appellant, leaves me satisfied that the Order made in "the Courts below was a correct and proper Order in the "circumstances.

"For these reasons I beg to move that this appeal should be "dismissed with costs."

The other members of the House were of the same opinion.

ACTUARIAL NOTE.

Extra Premiums on Limited Payment Policies.

GIVEN the extra premium for an ordinary whole-life policy (premiums payable throughout life), what is the corresponding extra for a limited payment policy ?

Dr. T. D. Lister, in his recent book on *Medical Examination for Life Assurance*, states (in effect) that the limited payment extra must clearly be "much larger" than the whole-life extra, because it is payable for a limited period only. The reviewer of the book in this *Journal* (vol. liii, p. 84) points out quite truly—so far as the comment goes—that the extra corresponding to the addition of a specified number of years to the age (*i.e.*, on the assumption that the distribution of extra risk is correctly represented by an increase of age) is in many cases greater on the whole-life table than on the limited payment table.

As the question is one that frequently arises in office-practice it may be useful to recall F. W. White's and W. J. H. Whittall's paper (*J.I.A.*, vol. xxiv, p. 385) in which are given *inter alia* specimen whole-life and limited payment extras by three hypothetical tables (representing respectively increasing, constant and decreasing distributions of extra risk) based on aggregate (H^M) mortality, and to supplement the results brought out in that paper by showing how the extras compare on certain hypotheses based on select mortality. Within the limits of practice the relative extras for different methods of assurance do not vary materially with the magnitude, as distinguished from the distribution, of the extra risk. If, for instance, a constant extra mortality be halved or doubled the proportionate decrease or increase in the limited payment (or endowment assurance) extra will be much the same as in the whole-life extra. It will be sufficient, therefore, to give specimen values of the ratios of the limited payment to the whole-life extras. The assumed hypotheses as to the distribution of extra risk are :

I. That $q'_{[x]+n} = q_{[x+5]+n}$ for all values of n, *i.e.*, that the extra risk is represented by an addition of 5 years to the age.

II. That a' at 3 per-cent $= a$ at 4 per-cent. This, as is well-known, gives an approximately constant addition to q.

III. That $q'_{[x]}=q_{[x]}+\cdot002$, $q'_{[x]+1}=q_{[x]+1}+\cdot004$, . . . $q'_{[x]+4}=q_{[x]+4}+\cdot01$ . . . . $q'_{65}=q_{65}+\cdot01$, $q'_{66}=q_{66}+\cdot008$, ... $q'_{70}=q_{70}$, *i.e.*, that the extra risk increases in A.P. during the first 5 years, remains constant thereafter up to age 65 and runs off in A.P. in the following 5 years.

IV. That $q'_{[x]+n}=q^{O^{[NM]}}_{[x]+n}$ for all values of n, *i.e.*, that the life is an $O^{[NM]}$ life.

Of these I and II are White and Whittall's I and II—the "increasing" and "constant" hypotheses—but on the basis of select mortality, III takes account of the practical considerations that the full extra risk does not develop for some years, since the life is select (although under-average) at entry, and that there is probably no extra risk in old-age, while IV applies to a class of cases familiar in office-experience although the precise nature of the extra risk does not admit of simple definition.

The resulting extras, *i.e.*, the differences between the 3 per-cent net premiums on the above-mentioned hypotheses and the $O^{[M]}$ 3 per-cent premiums, give the ratios shown in the following table:

Ratios of t-payment extras for age at entry x to whole-life extras.

Hypothesis	$t=10$			$t=20$			$t=30$	
	$x=30$	$x=40$	$x=50$	$x=30$	$x=40$	$x=50$	$x=30$	$x=40$
I	1·50	1·20	·93	·97	·87	·83	·87	·87
II	1·64	1·42	1·19	1·16	1·07	1·00	1·04	1.00
III	1·57	1·35	1·14	1·13	1.06	1·03	1·03	1·02
IV	1·80	1·51	1·24	1·19	1·07	·99	1·03	·98

Even if hypothesis I—which is probably very seldom applicable—be left out of account, it would appear that except when the number of payments is less than 20 the limited payment extra need not be materially larger than the whole-life extra.

Hypothetical distributions of extra risk, as a basis for the calculation of extras and debts, have been rather out of favour of late years. The argument seems to be that in the absence of suitable under-average experience tables all extras are more or less arbitrary, and that debts or limited payment or endowment assurance extras may be fixed arbitrarily with as much or as little reason as ordinary whole-life extras. But

this view leaves out of account (1) the relatively general consensus of opinion and uniformity of practice in regard to whole-life extras and (2) the desirability of consistency in the extras or debts quoted for different kinds of assurance on the same under-average life. As regards some of the more important impairments office-practice in the matter of whole-life extras has been determined directly or indirectly by statistical experience, and in other cases it is based as a rule on a more extensive general experience than is available for any other form of extra-rating. And whatever may be said for the arbitrary assessment of some one type of extra there are obvious objections to the quotation of several independent and arbitrary extras for the same risk; extras for different kinds of assurance ought to be consistent on some reasonable assumption as to the nature of the extra risk. There is also the further consideration that a hypothetical table is no worse than an under-average experience table which is not exactly suitable to the particular case—a consideration of some relevance in view of the fact that under-average tables could not possibly be constructed for more than a few main classes of impairment. In all the circumstances it is submitted that the whole-life (or long-term endowment assurance) extra as a standard, with the hypothetical distribution as a means of passing to other forms of extra-rating, affords at present the most satisfactory basis for office practice.

REVIEWS.

National Health Insurance: Valuations of Approved Societies as at 31 December 1918; *Report by the Government Actuary.*

[Cmd. No. 1662, London: H.M. Stationery Office 1922. Price 4*s*.]

This Report deals with 10,162 separate valuations embracing almost exactly 16¼ millions of members. Save for an insignificant number of delayed valuations (the existence of which any actuary acquainted with Friendly Societies would have presumed, even if the Report had not mentioned them) the great task has been completed, and it is possible to review the bases and the methods adopted as well as the results deduced.

The provisions of the Acts relating to National Health Insurance require that every Approved Society and every branch thereof shall be valued. These provisions were carried into effect by treating as

a valuation unit each portion of a society for which a separate benefit fund was kept. Thus, in some cases, males and females belonging to one society were valued together—in others, they were dealt with separately. Again, in some instances, a society with branches was valued as a whole, whereas, in others, each lodge was treated by itself. The valuations have been grouped according to the country (whether England, Scotland, Ireland or Wales) to which the unit was attributed. A society admitting members from more than one of these constituents of the then United Kingdom was valued as a whole (and included in the country in which its Head Office was situated) unless steps had been taken, in terms of the Act of 1913, to secure that the membership in each country should be treated as a separate society. It is important to remember these points in considering the results from the different classes of societies and those from the various countries.

The basis to be adopted in the valuation was not laid down by the Act of 1911 but, very wisely, was left to be prescribed. The Actuarial Advisory Committee, appointed in 1912, recommended that a uniform standard should be applied to all societies, irrespectively of special conditions such as a particularly select (or a relatively undesirable) class of members. In the report of the Committee--printed as Appendix B in the publication now under review—the reasons for this recommendation are set forth in detail. Briefly, they are based upon the view that the object of the reserve values to be credited at the inception of the scheme (in accordance with the original Act of 1911) was merely to remedy the consequences of inequality of age, and not to take account of the probable liabilities assumed by each individual society. Accordingly, the actual initial reserves had been calculated upon a general average basis; and if an attempt had been made, at the first valuation, to consider the special circumstances of a society, a large surplus or deficiency might have been disclosed. Most of this surplus or deficiency would have arisen owing to the change of basis and the actual working-result of the valuation period would have been obscured. Moreover, any deficiency brought out must have been made good, according to the original Act, within three years. It is true that the position had been altered somewhat by later Acts, but, none the less, it remained clear that it was impracticable to differentiate between the various societies in the matter of the basis of valuation.

The rates of sickness, disablement, mortality, &c., employed were, in effect, those used in the calculation of the initial reserve values. Thus, in valuing the future sickness and disablement benefits, the tables used were the Manchester Unity, Whole Society, 1893/7 table with a loading of approximately 12¾ per-cent in the case of men and of about 35 per-cent in the case of women.

The calculations were made upon the basis of a future rate of interest of 3 per-cent per annum in respect both of actually accumulated funds and of reserve values. At first sight this seems to be rather low, having regard to the freedom from income-tax

enjoyed by Approved Societies. One must remember, however, that, of the total funds belonging to all the societies on the valuation date (such total being almost £120,000,000) more than £64,000,000 consisted of the credit in respect of initial reserve values on which interest is allowed only at 3 per-cent. Of the remainder, a large percentage was represented by the Commissioners' Investment Account; and, according to the rules in force, interest on the sums at the credit of the various societies in this account was being allowed at the rate of 4 per-cent per annum. The balance of the income from the investments made by the Commissioners was being applied in the accumulation of an investment reserve. Similar arrangements were made with regard to another large sum (more than £21,500,000) described as Commissioners' Current Account. Apart from cash and miscellaneous assets, together reaching just over £1,000,000, there remain for consideration (in relation to the rate of interest) only the actual investments of the societies. These assets, consisting of a small sum in mortgages and of War Loan, War Bonds and other investments, amounted to less than £17,000,000, *i.e.*, to about 14 per-cent of the total funds. It appears therefore, that even allowing for the higher rates of interest secured on these investments, the assumption of 3 per cent per annum was appropriate.

Closely connected with the question of the rate of interest is that of the valuation of the assets. No depreciation arose in respect of the reserve values and, as regards the investments in the Commissioners' Investment Account and the money held by them on current account, it was considered that the margin of interest retained was sufficient to provide against loss. The power to vary the rate of interest credited and thus, within limits, to create a reserve in respect of sums invested by, or held by, the Commissioners, is a valuable safeguard against depreciation. The only other substantial group was the investments of the societies themselves. It was provided in the valuation regulations that these securities should be valued at cost price, and it is stated in the Report (p. 18) that "the element of depreciation had not up to the valuation date become a serious factor." In this connection, it is pertinent to refer (see p. 156) to Section 21 of the report of the Actuarial Advisory Committee which recommended that, before certifying any portion of the surplus to be disposable, the valuer should make a specific reserve for any depreciation.

Coming now to the summary of the valuation balance sheets, it appears that, apart from the contingencies funds (established under an amending Act and amounting to over £6,500,000) the net surplus was £17,192,968. Towards this fine result 9,755 valuations, covering almost 16 millions of members, contributed a total surplus of £17,273,887—an average of £1·08 per member. Only 407 valuations, dealing with about 300,000 members, revealed deficiencies, the total being £80,919 or £·25 per member. Inspection of the analyzed tables shows the following general results:

(*a*) As regards the separate countries, English societies had the highest average surplus per member (£1·26) while the Welsh societies had the lowest (£·61). The great International societies, with most of their membership in England, produced an average surplus of £1·05 per member.

The average deficiency per member was highest in Wales (£·44) and lowest—as might, we think, have been expected —in the International societies (£·05).

(*b*) As regards sex, the average surplus per member was distinctly greater in the case of valuations relating to men only (£1·42) than in the case of those relating to women only (£·94). The major part of the difference is considered, however, to have been due to the lower contributions paid, and the smaller benefits received, in respect of women as compared with men.

(*c*) As regards types of societies no very marked differences arose in the average surplus per member when men only were considered. The general average surplus, as already stated, was £1·42, while the highest was £1·65 (Employers' Provident Funds) and the lowest £1·25 (Trades Unions). When women only were considered, the range was wider. Against the general average of £·94, the highest was £1·34 (Friendly Societies without branches) and the lowest £·55 (Employers' Provident Funds). It should be noted, however, that only three small cases appeared in this last group. When men and women in common insurance were considered, the general average was £·95 against a highest result of £1·29 (Employers' Provident Funds) and a lowest of £·66 (Industrial Assurance and Collecting Societies). A supplementary table makes it clear that the percentage of women in the last-mentioned case was very large in comparison with that in all the other groups of this classification (save only the Employers' Provident Funds which, however, had comparatively few members).

Efforts were made to trace the sources from which the net surplus came and the report contains a most interesting discussion of the effects of the various influences which had been at work. Lack of space precludes any reference to more than one or two of the chief points brought out.

(*a*) As regards disablement benefit, the saving in comparison with the basic estimates was, in the case of men, over £3,500,000 which sum has been carried forward as part of the non-disposable surplus. In Sir Alfred Watson's words this "constitutes a valuable reserve, the need for which may still arise." In the case of women, the saving was only about £100,000 and it appears that, even when their claims for *sickness* benefit fell (as they did, for special reasons, during the War) to a very low level, the cost of *disablement* allowance was nearly equal to the expectation according to the M.U. basis, loaded as already mentioned. The conclusion is that, unless great care

be exercised in supervision, this matter will create a difficult problem for some of the societies.

(*b*) On the whole, mortality produced a large loss, the gains due to the excessive death-rate among the younger men in consequence of the War being far more than counterbalanced by the very light mortality at the higher ages. There appears, however, to be good reason to believe that, in numerous cases, members who had died had been returned as lapsed with the result that the surplus was reduced owing to the regulations with regard to transfer values (p. 39).

(*c*) The results, generally, were abnormal, partly on account of the special circumstances of the War years which have left their mark on all sections of the experience of the Approved Societies, and partly because of the inevitable adjustments due to the fact that the valuations dealt with the first few years of the operation of a great and, in many ways, an experimental scheme. Much of the surplus revealed may possibly be required in future years to meet as yet undisclosed effects of the War.

The considerations just mentioned have been given full weight in the determination of disposable surpluses. None the less, a total sum of more than £9,000,000 (about 55 per-cent of the related surpluses) was available for the purpose of providing extra benefits during a period of five years for over $14\frac{1}{4}$ million members. The maximum disposable surplus in any case has been fixed as the equivalent of an increase of 5*s.* a week in sickness benefit with 2*s.* 6*d.* a week in disablement benefit and 10*s.* in maternity benefit. More than 3,000,000 members appear in the group whose disposable surpluses would permit of additional benefits equal to one-fifth of this maximum scale, $2\frac{1}{2}$ millions more appear in the group which could grant extra benefits of one-half of the maximum, and the remainder are distributed over the other groups. The number of members who could adopt the full scale of additional benefits is over 600,000.

In one place Sir Alfred Watson refers to a figure as an "impressive sum." This same adjective can be applied most fittingly to the Report itself.

R. C. S.

A Correlation Formula. *By* J. F. Steffensen, Ph.D.

[Pp. 19. Extract from Skandinavisk Aktuarietidskrift. Uppsala, 1922.]

The simplest form of frequency function depending on two variables, when there is no correlation, is obtained by multiplying a frequency function depending on one of the variables only, by a frequency function depending on the other variable only. In order to introduce correlation, terms involving both variables must be included. In this paper it is suggested that $f(x,y)$ might be taken as $\mathrm{K} f_1(x+cy) f_2(y+\gamma x)$, where f_1 and f_2 are some form of

frequency curve—in the numerical example a Pearson Type III curve is used. The method of application indicated is interesting; it proceeds by using series found from the correlation table by the summation of rows, columns and diagonals, and by a process which gets from one term to the next by moving one step down and two to the right. These series can be utilized to give product moments. The formula suggested would probably be most suitable if there were only slight correlation, and as the example given exhibits high correlation it is not unduly flattering. We are some way yet from a really suitable skew frequency function (2 variables): those that have been suggested are open to one or more of the objections that they are empirical or have too limited a scope, or lead to negative frequency or do not show directly in the equation the coefficient of correlation implied. Dr. Steffensen has given an interesting study which is more successful in its arithmetical application than many similar attempts by other writers.

Frequency Arrays. *By* H. E. Soper.

[London: Cambridge University Press, 1922. Price 3*s.* 6*d.*]

This pamphlet of about 50 pages illustrates the use of logical symbols in the study of statistical distributions. It can be recommended to anyone who is interested in the theoretical work underlying modern statistical research. The mathematics are not difficult and the method has the merit of leading to some interesting short proofs of known propositions.

La misura delle grandezze ed il problema delle prove ripetute nel calcolo delle probabilità. *By* Silvio Minetola.

[Pp. 24. Extract from Giornale di Matematica Finanziaria. Turin, 1922.]

The underlying idea is to make more use of the range of the statistics than has been customary: thus a mean is found in the ordinary way from 100 cases spread over a range of 39 to be 17·44 and the author takes 17·44 ÷ 39 or ·447 as a function for use as a kind of "probability" in much the same way as we use a standard deviation. The objection is that range is subject to a large "probable error" and while occasionally we may be able to fix it from theoretical considerations it is usually impossible to do so in practical statistical work.

Depreciation. *By* V. H. VELEY, D.SC., F.R.S.

[Pp. ix + 82. London: Gee & Co. 1922. Price 7*s*. 6*d*.]

THIS book sets out four methods by which wasting assets may be written down: (1) the straight line method, *i.e.*, equal annual amounts are written off; (2) reduction by balance of cost, *i.e.*, r per-cent of balance where r does not vary from year to year; (3) sinking fund method, *i.e.*, h is written off the first year and $h(1+i)^n$ in the $(n+1)$th year; (4) annuity certain method, *i.e.*, calculating a level annuity. An actuary would probably expect (4) to be the same as (3) and this seems to be implied (p. 65), but in the example on pp. 73, &c., the principal outstanding by (4) is written down by the whole annuity and the method, therefore, becomes like (1). This is intentional, as the author says (p. 79): "The annuity method is the straight line method in another form with higher charges to revenue for a shorter period; if the interest is only a book entry, and not a payment in sovereigns, then the extinction of capital is simply the sinking fund method over again." (!). The book wants revision and correction and the algebra is so badly printed that it should be reset. This is unfortunate as the subject is interesting and important to many concerns. If the earning capacity of a wasting asset varies and it is wished to arrange depreciation so that an equal rate of profit is disclosed each year there is no inherent difficulty if the capacity can be properly estimated. In practice we have to add to this initial difficulty those connected with satisfying the Inland Revenue, of meeting the payments required for the service of a loan on the asset, and so on.

Tables of the incomplete Γ*-function. Computed by the Staff of the Department of Applied Statistics, University of London, University College. Edited by* KARL PEARSON, F.R.S.

[Pp. xxxi + 164. London: H.M. Stationery Office, 1922. Price £2. 2*s*. net.]

THE incomplete Γ-function, $\int_0^x e^{-x}x^p dx$, is not convenient, as it stands, either for tabulation or for computation (on account of the largeness of its value when p is large and the variation in the requisite range of x for different values of p), and the principal tables in this volume give the values, for the arguments u and p, of the modified function $\int_0^{u\sqrt{p+1}} e^{-x}x^p dx/\Gamma(p+1)$. The choice of this particular (and, at first sight, artificial-looking) function—to which the name $I(u,p)$ is given—was determined by statistical considerations. It answers the necessary purpose of lessening the difficulties of tabulation and (to some extent) of computation, since the division by $\Gamma(p+1)$ brings down the value of the integral to between 0 and 1 and the alteration

of the upper limit reduces considerably the range of u, but its special feature is that it is the probability-integral of the Type III curve of the Pearson frequency family. If in the equation to that curve (*Frequency curves and correlation*, p. 65) the origin be moved from the mode to the start of the curve by substitution of $x-a$ for x, it will be found that the ratio of the integral from 0 to x to the integral from 0 to ∞ can be expressed in the form $\int_0^{\frac{px}{a}} x^p e^{-x} dx / \Gamma(p+1)$, so that the result of entering the I (u, p) Tables with $px/a\sqrt{p+1}$ (or, since $a\sqrt{p+1}/p=\sigma$, x/σ) and p is to give the proportion of the total number of cases occurring with the frequency 0 to x; the areas of the curve are thus obtained more accurately than by the ordinary process of calculating ordinates and using a quadrature formula. At the same time the tables are practically tables of the incomplete Γ-function, since the result of entering them with $x/\sqrt{p+1}$ and p, and multiplying by $\Gamma(p+1)$, is $\int_0^x e^{-x} x^p dx$, and they consequently supply—in combination with Legendre's complete Γ-function table recently reproduced in *Tracts for Computers, No.* IV*, or the table in *Tables for Statisticians and Biometricians*—a valuable addition to existing aids to mathematical and statistical computation.

The work of constructing the tables, and of devising special methods of applying them in certain boundary-areas where interpolation (even with the 2nd and 4th central differences given in Tables I and II) fails to yield good results and in areas outside the limits of the Tables, appears from the Editor's Introduction—a most instructive revelation of the difficulties that may arise in the computation of mathematical tables—to have been very great. It has been in progress intermittently since 1905, and during the last ten years "hardly any worker in the laboratory . . . . . has failed to contribute in some way to the progress of 'Gamma'." No student of the Introduction will fail to recognize the felicity and appropriateness of the quotation prefixed to the tables: "A "painful work it is I'll assure you, and more than difficult, "wherein what toyle hath been taken, as no man thinketh, so no "man believeth, but he that hath made the triall."

It may be of interest to note that I (u, p) has an application—although, it must be admitted, a somewhat academic one—even within the limited range of the older actuarial science. For in terms of the Makeham constants $\bar{a}_x$ can be expressed in the form $[K\{1-I(\beta/\sqrt{\rho+2},\ \rho+1)\}-1]/\rho+1 \log_e c$, where

$$K=\beta^{-\rho+1}e^{\beta}\Gamma(\rho+2)\ ;\quad \beta=c^x\log_e 1/g\ ;\quad \text{and}\quad \rho+1=(\log_e s-\delta)/\log_e c.$$

Hence the $I(u,p)$ table renders it possible to calculate the value of $\bar{a}_x$—or, with suitable changes in β and ρ, $\bar{a}_{xyz}$. . . .—for any given values of the Makeham constants and at any given rate of interest. Thus the text-book table 3 per-cent value of $\bar{a}_{30\,.\,40\,.\,50}$ will be found to

* A table of log $\Gamma(p)$ for arguments beyond Legendre's range, namely, for $p=2$ to $p=1200$, has now been published in *Tracts for Computers*, No. VIII.

be $20{\cdot}775 - 9{\cdot}197\Gamma({\cdot}47285)\{1 - I({\cdot}22971, -{\cdot}52715)\}$, which, as the result of entering the Γ Table in *Tables for Statisticians, &c.*, and the auxiliary I' Table in the work under review, gives 11·286 approximately. This example illustrates the perversity of the practical case in falling in an awkward region of the table, and also the value of the auxiliary tables that have been computed to facilitate the calculation of I in such cases. The computer is fortunate if he escape the region $p = -{\cdot}75$ to $-1{\cdot}0$, u small, since here I covers more than half of its journey from 0 to 1 between $u = 0$ and $u = {\cdot}1$.

O^{M} *values of* D, N, S *and* **a** *for quinquennial ages and a series of rates of interest.* *By* THV. RICHARDT.

[Pp. 18. Extract from 20th Bulletin of the Permanent Congress Committee. 1921.]

THE tables in this tract are based on the values for ages 10, 15, 20 &c. of D_x, $\mathbf{a}_{x\overline{5}|}$ and $(I\mathbf{a})_{x\overline{5}|}$, the values of D being calculated in the ordinary way and those of **a** and I**a** by approximate formulas deduced from the author's *Formula for the transformation of mortality tables* * but apparently more directly obtainable by expanding **a** in powers of $i' - i$; thus the approximate formula used for calculating the differences of **a** corresponding to differences of ·0025 in the rate of interest seems to be (in effect)

$$\Delta\mathbf{a}^{i''}/\Delta\mathbf{a}^{i'} = (1 + {\cdot}00125r)/(1 - {\cdot}00125r),$$

where r denotes the ratio of the second to the first differential coefficient of $\mathbf{a}'$ according to i'.

While the author's method of obtaining **a** is interesting—and sufficiently accurate for a 4-year term annuity—it entails a preliminary calculation of the values of r at two or three rates of interest, and we are inclined to think that equally good results could be obtained with less work by calculating the values of $\mathbf{a}_{x\overline{5}|}$ from log vp at 6 per-cent and interpolating between these and the known values at 3 and 4 per-cent. But the calculation of the values of $\mathbf{a}_{x\overline{5}|}$ is merely a step in the process of calculating D_x N_x and $\mathbf{a}_x$, and the values of these quantities here tabulated for ages 10, 15, 20, &c., at various rates from $4\frac{1}{4}$ to $6\frac{1}{4}$ per-cent may occasionally be useful.

At the foot of page 6 a minus sign appears to have been incorrectly inserted in the expression for $\mathbf{a}(i'') - \mathbf{a}(i')$.

* Skandinavisk Aktuarietidskrift.

Francis Galton. 1822–1922. *A Centenary Appreciation.* *By* KARL PEARSON, F.R.S. *With frontispiece drawing of Francis Galton.*

[Pp. 23. London: Cambridge University Press. Price 2*s*. net.]

THE Galton Laboratory and the Galton Professor of the University of London have contributed so much to the advancement of

statistical science that actuarial students who are interested in modern statistical work will, we believe, be glad to read this centenary tribute to the founder of the science of Eugenics. (For Prof. Pearson holds that although someone may have thought of Eugenics before Galton, just as someone may have discovered evolution before Darwin or logarithms before Napier, "the name "and idea of a science of Eugenics have become world-wide only "since Galton made his appeal and showed its possibilities"). It is an appreciation, as the title indicates, not a biographical sketch, and it has little or nothing to say about the bare facts of Galton's life. But it gives what is more important and interesting, namely some idea of his mental development and of how the study of the Darwinian theory of evolution led him to the problem of the relative importance of nature and nurture and eventually to the Eugenic solution which "in the last years of his life . . . had grown to be a faith." Galton had in his nature something of the prophetic genius, but he was a very human prophet for he wrote a novel in the last year of his long life. "His mind was 'boyish' at his majority, youthful in the fifties and still that of a joyful pioneer at his death."

There could be no better tribute to Galton than the vigorous vitality of the work that is carried on under his name. Of that vitality this appreciation is a characteristic example. It is an authentic product of the author's pen, with its apposite quotation on the fly-leaf, its spirited defence of the much-despised mid-Victorian age (to the list of "some eminent Victorians" whom we already know Prof. Pearson adds the names of Darwin, Lyell, Maxwell and Kelvin), its castigation of some of Galton's critics, its chance *dictum*—which we find in a footnote—that "a glorified "slide-rule which will read to 8 or 12 figures is the mechanical "calculator of the future."

Complementi di Matematica. *By* PROF. S. O. CARBONI.

[Pp. viii + 400. Rome: Societá Editrice Athenæum, 1922. Price L. 60.]

THIS book is based on a course of lectures delivered by the author to the Higher Commercial Institute at Genoa. The list of subjects dealt with is rather suggestive of the new syllabus for Part I, and although in some respects it goes a good deal beyond and in others falls short of the scope of that examination some of the chapters, such as those on the elementary notions of analytical geometry and numerical and graphic calculation, might give useful hints. Some exercises and examples—not always indexed—will be found appended to, or interpolated in, the various chapters.

A feature of the book is the discussion of the application of the differential and integral calculus and other mathematical methods to the problems of economics, finance, vital statistics and life assurance. The fact seems worth mentioning as a more or less new departure, although the particular applications discussed appear to be elementary and do not call for special comment.

Versicherungswesen. *By* PROF. A. MANES, PH.D., &c. *Third Edition.*

[Vol. I, pp. xiv + 231; vol. II, pp. xiv + 357. Leipzig: B. G. Teubner, 1922.]

IN looking through this new edition of Prof. Manes's well-known handbook we notice a table (vol. ii, p. 9) giving, among other statistics of German life offices, the average sum assured per policy in force at quinquennial intervals from 1830 to 1920. The average in 1920 was 6,538 marks, which represents now less than 5*s*. No doubt new policies are being effected for much larger amounts, but there must be still on the books of the companies a very great number of policies that have become under present conditions of little value as life assurance cover. The fact shows in a striking way the dependence of such a business as insurance on the stability of the currency. Here is a vast superstructure of organized effort and thrift of which the very foundation is crumbling. Depreciation may not materially affect the financial position of the companies except as regards expense of management, since both liabilities and assets are mainly in currency, but on the scale it has reached on the Continent it strikes at the very root of the insurance idea. It seems hardly possible that the mark can recover its old value, but stabilization on almost any basis would be better for insurance business than the existing uncertainty.

Depreciation is only one, although the most important, of the effects of the war on insurance business. Prof. Manes has accordingly found much new matter to be dealt with since the publication of the second edition of the handbook in 1913—the subject of war claims and the issue of war-loan policies for instance (to mention only questions arising in connection with life assurance)—and the book is now issued in two volumes, the first dealing with the history, organization, and conduct of insurance business generally and the second in more detail with the several branches, life, fire, accident, sickness, &c. The whole work is characterized by the thoroughness and extensive knowledge of which the author's name is a guarantee.

ORIGINAL TABLES.

English Life Table No. 8. *Force of Mortality and* $3\frac{1}{2}$ *per-cent functions. Contributed by* CHAS. H. ASHLEY, A.I.A., *Actuary of the London and Manchester Assurance Company, Ltd.*

THE following tables have been computed primarily for the business of Industrial Assurance and the columns presented are the ones usually required. μ_x is wanted for calculating expected claims and to produce $\bar{a}_x$ values comparable with those based on the English Life Table No. 6 (*J.I.A.*, vol. xliii, p. 351) and No. 3 (*J.I.A.*, vol. xxxiv, p. 122) already published. The columns have been carried on as far as ages 106 (male) and 108 (female) as is done in the

official tables graduated by Mr. George King for the Registrar General (Cd. 7512), the fundamental basis being the official p_x.

The first 7 values of μ_x (either sex) were obtained from the formula

$$\mu_x = -\frac{1}{M}\left\{\delta\log_{10}l_x - \frac{1}{2}\delta^2\log_{10}l_x + \frac{1}{3}\delta^3\log_{10}l_x - ,\&c.\right\} \quad . \quad . \quad (1)$$

(where M is the modulus of common logarithms) as far as 8th differences for the first 4 ages (*e.g.*, males $\delta^8\log_{10}l_0 = +\cdot0046395$ and $\delta^8\log_{10}l_3 = +\cdot0004313$) 4th differences sufficing for μ_4, μ_5 and μ_6. The formula gives values gradually approaching, and at μ_6 coinciding with, the values to 5 decimal places yielded by the formula

$$\mu_x = \frac{7(d_{x-1}+d_x)-(d_{x-2}+d_{x+1})}{12l_x} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which was used to carry the table up to the final value (either sex). The formula

$$\Delta\mu_x = \left(\frac{1\cdot1 - p_x}{p_x}\right)\frac{d\,\log_e(q_x + \cdot1)}{dx} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

apparently used by Mr. Marr (*J.I.A.*, vol. xliii, p. 352), was tried for ages 95 and over, but, as none of its orders of differences are either constant or regular, it was not thought that the results produced were any more reliable than by formula (2) and in any case the modifications would have been relatively small. In applying formula (2) after age 90, the decimal figures were first restored to l_x: this has an appreciable effect. The value of μ_0 must be read cautiously, for it is probably of no practical use alone.

D_x was worked out arithmetically and checked on the "Millionaire" machine, 7 figure values of v^x being used, and effect being given when it became necessary towards the end of the table, to the decimal figures of l_x.

N_x was obtained by summing D_x; then a_x to 5 decimal places by dividing N_{x+1} by D_x.

$\bar{a}_x$ was calculated to 5 decimal places by the formula

$$\bar{a}_x = a_x + \frac{1}{2} - \frac{1}{12}(\mu_x + \delta)$$

from which $\bar{A}_x$ to 7 decimals (of necessity) was derived from

$$\bar{A}_x = 1 - \delta\bar{a}_x$$

and $\bar{P}_x$ to 5 decimals from

$$\bar{P}_x = \frac{1}{\bar{a}_x} - \delta \text{ and } \bar{P}_x = \frac{\bar{A}_x}{\bar{a}_x}$$

Then followed $\bar{N}_x$ from $\bar{a}_x \times D_x$ and $\bar{M}_x$ from $\bar{A}_x \times D_x$ and $\bar{C}_x$ by differencing $\bar{M}_x$.

All these calculations were done in duplicate independently, the machine being used for checking wherever possible.

Table I.

English Life Table No. 8 (Male).

x	μ_x	Interest $3\frac{1}{2}$ per-cent			x
		$\bar{a}_x$	$\bar{A}_x$	$\bar{P}_x$	
0	·23130	20·535	·29358	·01430	0
1	·06113	23·093	·20556	·00890	1
2	·01728	23·717	·18409	·00776	2
3	·01059	23·856	·17932	·00752	3
4	·00658	23·873	·17872	·00749	4
5	·00542	23·837	·17997	·00755	5
6	·00441	23·773	·18219	·00766	6
7	·00357	23·683	·18526	·00782	7
8	·00289	23·572	·18909	·00802	8
9	.00239	23·443	·19354	·00826	9
10	·00204	23·298	·19851	·00852	10
11	·00185	23·142	·20389	·00881	11
12	·00179	22·977	·20956	·00912	12
13	·00186	22·806	·21544	·00945	13
14	·00201	22·631	·22145	·00978	14
15	·00223	22·455	·22753	·01013	15
16	·00247	22·277	·23365	·01049	16
17	·00269	22·097	·23982	·01085	17
18	·00291	21·916	·24606	·01123	18
19	·00315	21·733	·25236	·01161	19
20	·00338	21·548	·25872	·01201	20
21	·00359	21·361	·26516	·01241	21
22	·00374	21·170	·27171	·01283	22
23	·00383	20·975	·27842	·01327	23
24	·00390	20·774	·28534	·01374	24
25	·00396	20·567	·29248	·01422	25
26	·00405	20·352	·29985	·01473	26
27	·00418	20·132	·30743	·01527	27
28	·00433	19·906	·31521	·01584	28
29	·00450	19·674	·32318	·01643	29
30	·00469	19·437	·33135	·01705	30
31	·00491	19·194	·33971	·01770	31
32	·00516	18·946	·34824	·01838	32
33	·00544	18·693	·35694	·01909	33
34	·00575	18·435	·36580	·01984	34
35	·00609	18·173	·37481	·02062	35
36	·00644	17·907	·38397	·02144	36
37	·00679	17·636	·39330	·02230	37
38	·00715	17·360	·40280	·02320	38
39	·00753	17·079	·41247	·02415	39
40	·00793	16·792	·42233	·02515	40
41	·00837	16·500	·43237	·02620	41
42	·00887	16·204	·44257	·02731	42
43	·00941	15·902	·45293	·02848	43
44	·00998	15·597	·46344	·02971	44
45	·01061	15·287	·47410	·03101	45
46	·01129	14·973	·48490	·03239	46
47	·01201	14·655	·49583	·03384	47
48	.01277	14·334	·50690	·03536	48
49	·01357	14·008	·51812	·03699	49

TABLE I—(*continued*).

x	μ_x	Interest 3½ per-cent $\bar{a}_x$	$\bar{A}_x$	$\bar{P}_x$	x
50	·01445	13·678	·52947	·03871	50
51	·01544	13·344	·54094	·04054	51
52	·01655	13·008	·55250	·04247	52
53	·01778	12·670	·56413	·04453	53
54	·01911	12·331	·57581	·04669	54
55	·02056	11·990	·58753	·04900	55
56	·02214	11·649	·59927	·05145	56
57	·02384	11·307	·61101	·05404	57
58	·02567	10·966	·62276	·05679	58
59	·02763	10·625	·63449	·05971	59
60	·02976	10·284	·64621	·06283	60
61	·03206	9·945	·65789	·06615	61
62	·03454	9·606	·66954	·06970	62
63	·03717	9·269	·68114	·07348	63
64	·03998	8·933	·69270	·07754	64
65	·04306	8·598	·70422	·08191	65
66	·04648	8·265	·71566	·08659	66
67	·05030	7·935	·72701	·09162	67
68	.05450	7·610	·73822	·09700	68
69	·05910	7·289	·74926	·10279	69
70	·06415	6·973	·76012	·10901	70
71	·06970	6·663	·77077	·11568	71
72	·07583	6·361	·78119	·12281	72
73	·08265	6·066	·79132	·13046	73
74	·09016	5·781	·80113	·13858	74
75	·09829	5·507	·81057	·14719	75
76	·10697	5·243	·81965	·15633	76
77	·11620	4·988	·82839	·16608	77
78	·12616	4·745	·83677	·17635	78
79	·13690	4·512	·84476	·18718	79
80	·14834	4·291	·85238	·19865	80
81	·16036	4·081	·85962	·21064	81
82	·17288	3·880	·86651	·22333	82
83	·18596	3·689	·87309	·23667	83
84	·19970	3·506	·87940	·25082	84
85	·21433	3·330	·88543	·26590	85
86	·22992	3·162	·89121	·28185	86
87	·24697	3·002	·89671	·29871	87
88	·26650	2·853	·90184	·31611	88
89	·28797	2·721	·90638	·33311	89
90	·30977	2·609	·91025	·34889	90
91	·33000	2·517	·91341	·36289	91
92	·34680	2·441	·91601	·37526	92
93	·35956	2·375	·91829	·38665	93
94	·36872	2·309	·92056	·39869	94
95	·37574	2·233	·92317	·41343	95
96	·38305	2·138	·92644	·43333	96
97	·39415	2·017	·93061	·46139	97
98	·41347	1·868	·93574	·50093	98
99	·44649	1·693	·94176	·55627	99
100	·49965	1·501	·94838	·63227	100
101	·58044	1·301	·95525	·73424	101
102	·69756	1·105	·96197	·87060	102
103	·86128	·923	·96824	1·04902	103
104	1·08338	·760	·97386	1·28139	104
105	1·37298	·617	·97876	1·58635	105
106	1·70906	·491	·98309	2·0003	106

Table II.

English Life Table No. 8 (*Male*). $3\frac{1}{2}$ *per-cent.*

x	D_x	$\bar{N}_x$	$\bar{C}_x$	$\bar{M}_x$	x
0	1000000	20534510	118896	293584	0
1	849815	19624982	28708	174688	1
2	792965	18806941	10433	145980	2
3	755910	18033087	6088	135547	3
4	724373	17293304	4250	129459	4
5	695702	16583428	3344	125209	5
6	668891	15901285	2610	121865	6
7	643708	15245110	2033	119255	7
8	619943	14613389	1600	117222	8
9	597407	14004809	1292	115622	9
10	575936	13418220	1095	114330	10
11	555384	12852635	987	113235	11
12	535632	12307184	957	112248	12
13	516579	11781137	977	111291	13
14	498149	11273824	1035	110314	14
15	480286	10784654	1108	109279	15
16	462955	10313077	1178	108171	16
17	446142	9858569	1223	106993	17
18	429852	9420615	1278	105770	18
19	414060	8998700	1327	104492	19
20	398753	8592334	1364	103165	20
21	383928	8201032	1381	101801	21
22	369587	7824316	1375	100420	22
23	355738	7461693	1351	99045	23
24	342380	7112677	1321	97694	24
25	329504	6776774	1294	96373	25
26	317089	6453516	1279	95079	26
27	305109	6142448	1274	93800	27
28	293539	5843158	1271	92526	28
29	282363	5555241	1271	91255	29
30	271565	5278308	1277	89984	30
31	261126	5011990	1288	88707	31
32	251030	4755942	1303	87419	32
33	241260	4509823	1324	86116	33
34	231800	4273316	1344	84792	34
35	222640	4046124	1367	83448	35
36	213768	3827944	1385	82081	36
37	205177	3618493	1401	80696	37
38	196862	3417497	1415	79295	38
39	188814	3224681	1427	77880	39
40	181026	3039781	1444	76453	40
41	173485	2862548	1462	75009	41
42	166181	2692732	1484	73547	42
43	159102	2530110	1509	72063	43
44	152239	2374456	1532	70554	44
45	145584	2225562	1558	69022	45
46	139130	2083221	1584	67464	46
47	132868	1947238	1608	65880	47
48	126794	1817422	1630	64272	48
49	120904	1693588	1652	62642	49

TABLE II—(*continued*).

x	D_x	$\overline{N}_x$	$\overline{C}_x$	$\overline{M}_x$	x
50	115191	1575556	1678	60990	50
51	109646	1463151	1710	59312	51
52	104258	1356211	1742	57602	52
53	99019	1254585	1778	55860	53
54	93923	1158126	1813	54082	54
55	88965	1066693	1846	52269	55
56	84142	980151	1878	50423	56
57	79450	898366	1909	48545	57
58	74887	821207	1934	46636	58
59	70453	748548	1958	44702	59
60	66146	680259	1978	42744	60
61	61965	616214	1993	40766	61
62	57910	556287	2004	38773	62
63	53982	500352	2007	36769	63
64	50183	448280	2005	34762	64
65	46516	399941	2001	32757	65
66	42976	355207	1993	30756	66
67	39563	313947	1984	28763	67
68	36275	276039	1968	26779	68
69	33114	241355	1944	24811	69
70	30083	209768	1914	22867	70
71	27185	181144	1873	20953	71
72	24424	155352	1828	19080	72
73	21802	132250	1773	17252	73
74	19322	111700	1707	15479	74
75	16990	93557	1629	13772	75
76	14815	77668	1536	12143	76
77	12804	63872	1436	10607	77
78	10960	52004	1327·8	9171·0	78
79	9284·5	41896	1212·9	7843·2	79
80	7778·6	33379	1093·6	6630·3	80
81	6440·9	26283	971·8	5536·7	81
82	5268·1	20442	850·6	4564·9	82
83	4254·2	15694	733·4	3714·3	83
84	3389·7	11884	623·1	2980·9	84
85	2662·9	8868·2	521·5	2357·8	85
86	2060·5	6516·3	429·5	1836·3	86
87	1568·8	4710·3	349·0	1406·8	87
88	1172·9	3346·8	279·36	1057·8	88
89	858·85	2337·3	218·28	778·44	89
90	615·39	1605·6	165·85	560·16	90
91	431·69	1086·6	122·01	394·31	91
92	297·27	725·77	87·07	272·30	92
93	201·71	479·10	60·59	185·23	93
94	135·39	312·63	41·417	124·64	94
95	90·149	201·32	28·000	83·223	95
96	59·608	127·45	18·872	55·223	96
97	39·062	78·796	12·751	36·351	97
98	25·221	47·108	8·654	23·600	98
99	15·870	26·868	5·8709	14·946	99
100	9·5690	14·358	3·9178	9·0751	100
101	5·3989	7·0237	2·5014	5·1573	101
102	2·7609	3·0520	1·4665	2·6559	102
103	1·2284	1·1341	·74924	1·1894	103
104	·45198	·34350	·31364	·44016	104
105	·12927	·07981	·10028	·12652	105
106	·02669	·01312	—	·02624	106

Table III.

English Life Table No. 8 (Female).

x	μ_x	Interest 3½ per-cent			x
		$\bar{a}_x$	$\bar{A}_x$	$\bar{P}_x$	
0	·17421	21·388	·26423	·01235	0
1	·05463	23·471	·19255	·00821	1
2	·01729	24·063	·17220	·00715	2
3	·01037	24·215	·16696	·00689	3
4	·00645	24·244	·16597	·00685	4
5	·00533	24·220	·16679	·00689	5
6	·00433	24·169	·16855	·00697	6
7	·00351	24·094	·17115	·00710	7
8	·00286	23·997	·17447	·00727	8
9	·00238	23·883	·17839	·00747	9
10	·00206	23·755	·18280	·00770	10
11	·00189	23·616	·18757	·00794	11
12	·00186	23·470	·19260	·00820	12
13	·00193	23·319	·19780	·00848	13
14	·00209	23·165	·20309	·00877	14
15	·00229	23·010	·20843	·00906	15
16	·00249	22·854	·21381	·00935	16
17	·00264	22·696	·21924	·00966	17
18	·00273	22·534	·22479	·00997	18
19	·00282	22·369	·23047	·01030	19
20	·00291	22·200	·23630	·01064	20
21	·00300	22·026	·24229	·01100	21
22	·00309	21·847	·24843	·01138	22
23	·00318	21·664	·25474	·01176	23
24	·00326	21·475	·26123	·01216	24
25	·00335	21·281	·26790	·01259	25
26	·00346	21·082	·27475	·01303	26
27	·00358	20·878	·28178	·01349	27
28	·00372	20·668	·28899	·01398	28
29	·00387	20·453	·29638	·01449	29
30	·00403	20·233	·30394	·01502	30
31	·00421	20·009	·31168	·01557	31
32	·00441	19·779	·31958	·01615	32
33	·00463	19·544	·32766	·01677	33
34	·00486	19·304	·33591	·01740	34
35	·00511	19·060	·34432	·01806	35
36	·00537	18·810	·35290	·01876	36
37	·00563	18·556	·36165	·01949	37
38	·00590	18·296	·37059	·02025	38
39	·00617	18·031	·37973	·02106	39
40	·00646	17·759	·38906	·02190	40
41	·00679	17·482	·39860	·02280	41
42	·00714	17·199	·40832	·02374	42
43	·00751	16·911	·41824	·02474	43
44	·00791	16·617	·42836	·02578	44
45	·00834	16·317	·43867	·02688	45
46	·00882	16·012	·44917	·02805	46
47	·00934	15·701	·45986	·02929	47
48	·00988	15·385	·47072	·03060	48
49	·01047	15·064	·48178	·03198	49

TABLE III—(*continued*).

x	μ_x	Interest 3½ per-cent $\bar{a}_x$	$\bar{A}_x$	$\bar{P}_x$	x
50	·01111	14·737	·49302	·03346	50
51	·01184	14·405	·50443	·03501	51
52	·01267	14·070	·51599	·03667	52
53	·01359	13·730	·52766	·03843	53
54	·01460	13·388	·53945	·04029	54
55	·01569	13·042	·55133	·04228	55
56	·01688	12·694	·56329	·04438	56
57	·01816	12·344	·57534	·04661	57
58	·01951	11·992	·58746	·04899	58
59	·02095	11·637	·59966	·05153	59
60	·02252	11·280	·61194	·05425	60
61	·02425	10·922	·62428	·05715	61
62	·02614	10·562	·63666	·06028	62
63	·02812	10·200	·64910	·06364	63
64	·03025	9·837	·66159	·06725	64
65	·03264	9·472	·67414	·07118	65
66	·03542	9·108	·68668	·07539	66
67	·03870	8·745	·69916	·07995	67
68	·04251	8·387	·71148	·08483	68
69	·04680	8·036	·72357	·09006	69
70	·05151	7·692	·73538	·09561	70
71	·05659	7·358	·74689	·10152	71
72	·06196	7·031	·75812	·10783	72
73	·06769	6·712	·76909	·11459	73
74	·07388	6·401	·77978	·12180	74
75	·08061	6·099	·79019	·12956	75
76	·08798	5·805	·80030	·13786	76
77	·09621	5·520	·81009	·14676	77
78	·10555	5·248	·81947	·15614	78
79	·11589	4·990	·82835	·16600	79
80	·12693	4·748	·83668	·17621	80
81	·13827	4·521	·84447	·18679	81
82	·14953	4·308	·85180	·19773	82
83	·16070	4·105	·85879	·20921	83
84	·17211	3·909	·86552	·22142	84
85	·18404	3·719	·87206	·23449	85
86	·19695	3·534	·87843	·24856	86
87	·21127	3·354	·88461	·26375	87
88	·22712	3·182	·89052	·27986	88
89	·24431	3·019	·89613	·29683	89
90	·26264	2·864	·90146	·31476	90
91	·28172	2·721	·90639	·33311	91
92	·30121	2·585	·91107	·35245	92
93	·32108	2·455	·91553	·37293	93
94	·34155	2·330	·91985	·39478	94
95	·36305	2·206	·92410	·41891	95
96	·38626	2·083	·92835	·44568	96
97	·41210	1·958	·93264	·47632	97
98	·44175	1·831	·93700	·51175	98
99	·47658	1·702	·94144	·55314	99
100	·51825	1·572	·94593	·60173	100
101	·56863	1·441	·95043	·65956	101
102	·62982	1·311	·95491	·72838	102
103	·70415	1·184	·95928	·81021	103
104	·79407	1·061	·96349	·90811	104
105	·90202	·945	·96750	1·02380	105
106	1·02987	·834	·97130	1·16464	106
107	1·17754	·727	·97499	1·34112	107
108	1·33922	·606	·97916	1·61655	108

TABLE IV.

English Life Table No. 8 (Female). 3½ per-cent.

x	D_x	$\bar{N}_x$	$\bar{C}_x$	$\bar{M}_x$	x
0	1000000	21387700	96362	264233	0
1	871816	20462698	27454	167871	1
2	815442	19622022	10611	140417	2
3	777453	18826188	6138	129806	3
4	745138	18065260	4292	123668	4
5	715724	17334993	3379	119376	5
6	688202	16633175	2639	115997	6
7	662337	15958036	2061	113358	7
8	637914	15308022	1631	111297	8
9	614740	14681799	1331	109666	9
10	592644	14078187	1142	108335	10
11	571481	13496198	1047	107193	11
12	551127	12934956	1020	106146	12
13	531487	12393703	1047	105126	13
14	512485	11871771	1100	104079	14
15	494073	11368536	1159	102979	15
16	476225	10883437	1203	101820	16
17	458939	10415901	1209	100617	17
18	442231	9965366	1206	99408	18
19	426090	9531250	1199	98202	19
20	410503	9112998	1190	97003	20
21	395451	8710065	1182	95813	21
22	380917	8321924	1172	94631	22
23	366884	7948065	1158	93459	23
24	353339	7587994	1145	92301	24
25	340264	7241230	1137	91156	25
26	327640	6907313	1131	90019	26
27	315449	6585809	1129	88888	27
28	303672	6276278	1130	87759	28
29	292292	5978330	1132	86629	29
30	281295	5691568	1137	85497	30
31	270665	5415617	1144	84360	31
32	260387	5150122	1154	83216	32
33	250448	4894730	1164	82062	33
34	240834	4649117	1177	80898	34
35	231533	4412959	1190	79721	35
36	222534	4185951	1200	78531	36
37	213829	3967792	1207	77331	37
38	205411	3758196	1216	76124	38
39	197270	3556879	1220	74908	39
40	189399	3363565	1228	73688	40
41	181787	3177993	1240	72460	41
42	174421	2999910	1251	71220	42
43	167293	2829074	1262	69969	43
44	160395	2665247	1276	68707	44
45	153717	2508210	1290	67431	45
46	147251	2357742	1307	66141	46
47	140986	2213641	1326	64834	47
48	134915	2075705	1342	63508	48
49	129034	1943746	1360	62166	49

TABLE IV—(*continued*).

x	D_x	$\bar{N}_x$	$\bar{C}_x$	$\bar{M}_x$	x
50	123333	1817578	1382	60806	50
51	117804	1697023	1409	59424	51
52	112435	1581917	1440	58015	52
53	107217	1472102	1475	56575	53
54	102142	1367435	1508	55100	54
55	97205	1267773	1543	53592	55
56	92401	1172981	1576	52049	56
57	87727	1082928	1609	50473	57
58	83179	997485	1636	48864	58
59	78758	916527	1663	47228	59
60	74460	839928	1690	45565	60
61	70280	767567	1719	43875	61
62	66214	699330	1741	42156	62
63	62263	635101	1759	40415	63
64	58429	574764	1776	38656	64
65	54707	518205	1798	36880	65
66	51090	465315	1824	35082	66
67	47569	415994	1858	33258	67
68	44134	370149	1892	31400	68
69	40781	327699	1922	29508	69
70	37513	288559	1939	27586	70
71	34338	252642	1942	25647	71
72	31268	219848	1928	23705	72
73	28316	190066	1900	21777	73
74	25490	163174	1862	19877	74
75	22798	139042	1811	18015	75
76	20247	117531	1749	16204	76
77	17843	98499	1682	14455	77
78	15587	81796	1604	12773	78
79	13483	67274	1515·4	11169	79
80	11538	54777	1408·9	9653·6	80
81	9763·2	44141	1286·8	8244·7	81
82	8168·5	35191	1153·9	6957·9	82
83	6758·4	27742	1018·5	5804·0	83
84	5529·0	21613	886·5	4785·5	84
85	4471·0	16627	762·2	3899·0	85
86	3570·9	12619	647·9	3136·8	86
87	2813·6	9437·8	544·4	2488·9	87
88	2183·6	6949·2	450·7	1944·5	88
89	1666·9	5032·8	366·5	1493·8	89
90	1250·5	3581·9	293·43	1127·3	90
91	919·99	2503·3	228·76	833·87	91
92	664·17	1716·9	174·66	605·11	92
93	470·16	1154·4	130·41	430·45	93
94	326·18	759·93	95·25	300·04	94
95	221·61	488·92	68·10	204·79	95
96	147·24	306·67	47·658	136·69	96
97	95·463	186·93	32·620	89·032	97
98	60·205	110·26	21·797	56·412	98
99	36·768	62·694	14·170	34·615	99
100	21·614	33·973	8·911	20·445	100
101	12·135	17·484	5·3802	11·534	101
102	6·4444	8·4475	3·0856	6·1538	102
103	3·1984	3·7860	1·6595	3·0682	103
104	1·4621	1·5517	·82315	1·4087	104
105	·60522	·57170	·36962	·58555	105
106	·22231	·18545	·14691	·21593	106
107	·07079	·05147	·05039	·06902	107
108	·01903	·01153	—	·01863	108

Table V.

Ratios of Continuous Annuities. English Life No. 8 *to No.* 6 *and No.* 8 *to No.* 3 (*Male*). $3\frac{1}{2}$ *per-cent.*

Age	$\frac{\text{No. 8}}{\text{No. 3}}$	$\frac{\text{No 8}}{\text{No. 6}}$
10	1·080	1·034
20	1·084	1·045
30	1·071	1·054
40	1·052	1·058
50	1·033	1·056
60	1·017	1·054
70	1·009	1·053
80	·992	1·101
90	·995	1·210
100	·934	1·662

Table V indicates that a straightforward pure premium valuation by No. 8 will produce a higher net liability than by No. 3, but lower than by No. 6. This has been confirmed by trial, though the point is not of much practical importance because Industrial businesses (with one exception) are not valued on a true-age pure premium basis.

In a valuation based on net office premiums (the most usual form) the author is able to state from experiment that Nos. 6 and 8 both yield substantially higher positive reserves than No. 3, provided the percentage of the office premiums valued is common and is not greatly in excess of the percentage which the true pure premiums are of the office premiums.

THE INSTITUTE OF ACTUARIES.

EXAMINATIONS, APRIL 1922.

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EXAMINATION FOR ADMISSION TO THE CLASS OF ASSOCIATE (PART I.—SECTION A).

First Paper.

1. A and B are two villages 20 miles apart connected by a mountain pass with its summit equidistant from each. A tourist leaves A at 10 a.m., rests an hour at the summit and arrives at B at 6 p.m. Another tourist sets out from A $1\frac{1}{2}$ hours after the first, passes him just as he commences the descent, rests half-an-hour at B and then returns and meets him two miles from B. What was the second tourist's time for the 20 miles? Assume that each tourist walks at a uniform rate towards the summit and also at another uniform rate from the summit.

2. Prove that

$$(b+c-2a)^5+(c+a-2b)^5+(a+b-2c)^5$$
$$=15(b+c-2a)(c+a-2b)(a+b-2c)(\Sigma a^2-\Sigma bc).$$

3. Find the sum of n terms of the series

$$\{a(a+b)\ldots(a+\overline{r-1}b)\}+\{(a+b)(a+2b)\ldots(a+rb)\}+\ldots$$
$$+\{(a+\overline{n-1}b)(a+nb)\ldots(a+\overline{n+r-2}b)\}$$

Hence find the sum of n terms of the series

$$1\cdot3^3+3\cdot5^3+5\cdot7^3+\ldots$$

4. If $u_n=n(1+\kappa)u_{n-1}-n(n-1)\kappa u_{n-2}$ and $u_2=2u_1\kappa$ find the relation between u_n and u_1.

5. Prove that if the two infinite series

$$a_0 + a_1x + a_2x^2 + \ldots\ldots$$
$$b_0 + b_1x + b_2x^2 + \ldots\ldots$$

be equal to one another for all values of x for which they are convergent then will $a_0 = b_0$, $a_1 = b_1$, &c.

Expand $\dfrac{\log(1+x^2)}{e^x - 1}$ in ascending powers of x as far as the term involving x^5.

6. Prove that if u_x (a rational integral function of x) be divided by $x^2 - x(a+b) + ab$ the remainder will be

$$\frac{u_a - u_b}{a-b}x + \frac{au_b - bu_a}{a-b}$$

7. If x be one of the first hundred numbers chosen at random find the probability that $x + \dfrac{100}{x}$ is greater than 50.

8. If a die is thrown five times what is the probability that a six appears on at least two consecutive occasions?

9. Three posts are filled one after another by lot from amongst ten persons (A, B, C, D, and six others). The first by any one of the ten, the second by anyone except A and B, the third by anyone except C and D. What is the chance that A or B or both of them and C or D or both of them are chosen? No one can hold more than one post, and in drawing for the second and third posts the barred persons and any person previously chosen are excluded.

Second Paper.

1. Given $\sum\limits_1^{10} u_x = 500{,}426$, $\sum\limits_4^{10} u_x = 329{,}240$, $\sum\limits_7^{10} u_x = 175{,}212$, and $u_{10} = 40{,}365$. Find u_1.

2. Show that (1) $\Delta^n 0^m = n(\Delta^{n-1}0^{m-1} + \Delta^n 0^{m-1})$,

(2) $\Delta 0^m - \dfrac{\Delta^2}{2} 0^m + \dfrac{\Delta^3}{3} 0^m \ldots = 0$ when $m > 1$.

3. Prove that $\dfrac{du_x}{dx} = \Delta u_{x-\frac{1}{2}} - \dfrac{1}{24}\Delta^3 u_{x-\frac{3}{2}}$ approximately.

4. Prove Taylor's theorem for the expansion of $f(x+h)$ in a convergent series of ascending powers of h assuming such expansion possible.

Find the expansion of $\log(1+e^x)$ in ascending powers of x as far as the term containing x^4.

5. A function of x assumes the form $\frac{0}{0}$ on substitution of a given quantity a for x. Show how to evaluate it.

Find (1) $\mathrm{L}t_{x=a}\dfrac{x-a+\sqrt{x^m-a^m}}{\sqrt{x^n-a^n}}$

(2) $\mathrm{L}t_{x=0}(1+x^3)^{\frac{1}{x^2}}$

6. If $y=x^n(\log x)^2$ prove that

$$x^2\frac{d^{n+2}y}{dx^{n+2}}+x\frac{d^{n+1}y}{dx^{n+1}}=2\underline{|n}.$$

7. Find the integrals

$$(1)\ \int\frac{x^3}{\sqrt{x-1}}dx \qquad (2)\ \int\frac{x^2}{x^6-a^6}dx \qquad (3)\ \int\frac{\log(\log x)}{x}dx$$

8. Prove that

$$\int_a^b u_x v_x dx = u_a\int_a^b v_x dx + \int_a^b \frac{du_x}{dx}\left(\int_x^b v_x dx\right)dx$$

provided u_x, v_x, and their derivatives are finite and continuous between the limits a and b.

9. Find the limit when x is increased indefinitely of

$$\frac{1}{x}+\frac{1}{x+m}+\frac{1}{x+2m}+\ldots+\frac{1}{x+xm}$$

Examination for Admission to the Class of Associate (Part I.—Section B).

"A Short Collection of Actuarial Tables" and the Supplement will be supplied for use in answering this Paper.

1. Into a fund £1,000 is paid at the end of each year, and out of it at the commencement of each year after the first a sum beginning at £200 and increasing £100 per annum. If the fund starts at £5,000 and earns interest at the rate of 5 per-cent per annum, for how many years can the outgoings be met?

2. A company makes an issue of 1,000 bonds of £100 each, bearing interest at 7 per-cent per annum, payable half-yearly and redeemable at par in five years by a cumulative sinking fund with annual drawings. Find the amount to be set aside each year for the service of the loan and construct a schedule showing the interest and drawings for each year.

3. An advance of £10,000 is being repaid by an annuity for 30 years, payable yearly, calculated on a 4 per-cent basis. After the fifth year's payment has been made the borrower, being in difficulties, proposes to liquidate the amount outstanding by annual payments equal for the next five years to interest only at the rate of 4 per-cent per annum on the amount outstanding and thereafter increasing each year by a uniform amount sufficient to redeem the principal by the end of the original term of 30 years. What will be the amount of the annual increase?

4. An issue of bonds bearing interest at 6 per-cent per annum, payable half-yearly, redeemable in 10 years by equal annual drawings, bonds drawn in the nth year being repayable at a premium of $\frac{1}{2}n$ per-cent, is offered at a price to yield 7 per-cent per annum convertible half-yearly. Find the price of issue.

5. A government issues at 80 a 4 per-cent loan for £100,000, interest payable yearly, and makes provision for redemption at 102 by annual drawings by setting aside a cumulative sinking fund of 1 per-cent per annum.

In how many years will the loan be redeemed, and what is the rate of interest realized by a syndicate taking up the whole loan, if they are allowed a commission of 2 per-cent on the nominal amount of the loan?

6. A company issued a 25-year leasehold redemption policy for £10,000 five years ago. If the assured now wishes to extend the term of the assurance by 10 years, what would be the new annual premium commencing immediately, assuming interest at 3 per-cent per annum throughout?

Examination for Admission to the Class of Associate (Part II).

First Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Give expressions in terms of d and l for the probabilities that in the nth year from the present time, of three lives aged respectively x, y and z,

(*a*) (x) will die and will be the second life to fail:

(*b*) the second life will fail:

(*c*) one at least will die.

2. (*a*) Find an expression for calculating the probability that of three lives aged x, y and z respectively, (x) will die first within n years of the third death.

(*b*) Find an expression for the probability that of the same three lives, (x) will die first and (z) will die last and within n years of the death of (x).

3. A municipality containing a stationary population decides to raise a forced loan by issuing to every inhabitant now between the exact ages of 30 and 35, a non-transferable bond for £100—redeemable at the end of five years or at the end of the year of previous death of the holder. Interest of £4 yearly is to be paid on each bond until redeemed.

At what uniform price must the bonds be issued if the rate of interest involved to the municipality on the whole issue is to be 5 per-cent.

Assume the population to be subject to H^M Mortality.

4. Given that at 3 per-cent $a_{30} = 19{\cdot}895$ and at 4 per-cent $a_{30} = 17{\cdot}155$, find as accurately as possible with the aid of these values and tables of annuities-certain the value of a_{30} at $3\frac{1}{2}$ per-cent.

5. (*a*) Which is the greater, $a_{\overline{\mathring{e}_x}|}$ or a_x? Prove your answer.

(*b*) Find an approximate value for ϕ such that $a_x = a_{\overline{\mathring{e}_x + \phi}|}$.

6. (i) Find (*a*) the most probable number of deaths and (*b*) the expected amount of the claims, that will occur in one year amongst m lives aged x at the beginning of the year each insured for a sum of 1.

(ii) A life office has the following whole-life assurances on its books, the sum assured in each case being £100, namely:

750 assurances on lives aged 30; 755 on lives aged 31; 761 on lives aged 32; and 764 on lives aged 33.

On the basis of the H^M Table, find the most probable number of deaths in one year, and the expected amount of claims during the year.

7. Given the following values:

A_{40} at $3\% = {\cdot}4706$ $\quad$ A_{40} at $3\frac{1}{2}\% = {\cdot}4216$ $\quad$ A_{40} at $4\% = {\cdot}3794$,
A_{40} at $4\frac{1}{2}\% = {\cdot}3430$ $\quad$ A_{40} at $5\% = {\cdot}3114$ $\quad$ D_{40} at $3\% = 25223$,

find an approximate value for R_{40} at 3%.

Second Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Find by the H^M Table at 3 per-cent the net annual premium, payable throughout the duration of the contract, for an assurance of £100, payable at the end of the year of the second death among four lives aged 30, 35, 40 and 45 years respectively.

2. Find an expression for the annual premium, payable throughout the duration of the contract, for an assurance payable in the event of two lives aged 30 and 35 respectively, both dying before reaching age 50, and in the lifetime of a third person now aged 45.

3. Find the value of an annuity of 1 per annum payable annually, the first payment being deferred until the end of the year in which the failure of the joint lifetime of (x) and (y) occurs, to be payable thereafter for a term of 20 years or until 10 years after the death of the survivor, whichever be the longer period.

4. Obtain the value, by the Carlisle Table at 3 per-cent, of a continuous annuity of £100 per annum to be entered upon at the death of a person aged 40 if he die within ten years after the death of a person aged 45, the annuity to cease on the expiration of a period of ten years from the death of (45). Assume ${}_{10}p_{40} = \cdot 86640$.

5. Find by the H^M Table at 3 per-cent the net annual premium for a whole-life assurance on a life aged 55, the sum payable at death being:

(*a*) £100 with a guaranteed simple reversionary bonus of 1 in respect of each premium paid vesting immediately; or

(*b*) the amount of the net premiums paid, without interest, whichever be the greater.

6. Employing the H^M values of μ_x and a_x, find an approximate value of $a_{30:40}$, assuming that tables of joint-life annuities are not available.

7. State concisely what is meant by the principle of uniform seniority and give an example of its applicability.

Discuss whether, and if so to what extent, the principle applies when $\mu_x = ma^x + nb^x$.

Third Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Find by the H^M Table at 3 per-cent the quarterly premium for a double endowment assurance on a life aged 35 for a term of twenty years, any unpaid instalments of the current year's premium to be deducted from the sum assured in the event of death before maturity of the policy.

Give expressions for the value of the policy at the end of 5 years, 5 years and 6 months, 5 years and 8 months, respectively.

2. (*a*) Under a child's deferred assurance £1,000 is payable at the death after age 21 of a child aged one at entry. Assuming H^M Mortality and interest at 3 per-cent, calculate what percentage loading is included in an office premium of £7 10*s*. In the event of the child dying during his minority the net premiums with compound interest at 3 per-cent are to be returned and mortality under age 21 can accordingly be ignored.

(*b*) In the above policy the following options are given at age 21:

(i) the policy may be surrendered for a cash payment equal to the net premiums accumulated at 3 per-cent compound interest.

(ii) the premiums may be discontinued and a paid-up policy exchanged securing a reduced sum assured at death; or

(iii) the policy may be converted into an endowment assurance payable at age 60 or earlier death, the net premium remaining unaltered.

Calculate the cash surrender value under (i) and the amounts assured under (ii) and (iii) respectively.

3. A life aged 30 effected a whole-life assurance, the number of annual premiums being limited to a maximum of 30. At the end of five years he ceased payment of premiums, the sixth being unpaid, and was granted a paid-up term assurance for the original sum assured for such a period as the value of the policy secured. Using the $O^{[NM]}$ 3 per cent Table, find the term of the paid-up assurance.

4. An endowment assurance for a term of 20 years is effected on a select life aged x. The sum assured is £100 with a guaranteed reversionary bonus of £2 in respect of each yearly premium paid, and the net premium is to increase annually for ten years by 10 per-cent of the first premium, the eleventh and subsequent premiums being of uniform amount.

On the basis of select tables find expressions for the initial and final net annual premiums.

Also give expressions for the value of the policy at the end of 12 years by the prospective and retrospective methods respectively and prove the two expressions equal.

5. Given a mortality table in which the rate of mortality is a function of both age and duration for the first three years after entry into insurance, but a function of the age alone thereafter, state how you would prepare tables of annuity values, (*a*) in respect of persons who have been insured three years or more, (*b*) in respect of persons at entry into insurance.

Given a_{63} at 3 % = 9·533, $q_{[60]} = \cdot01658$, $q_{[60]+1} = \cdot02173$ and $q_{[60]+2} = \cdot02732$, find the value of $a_{[60]}$.

6. Given Makeham's Law, $\mu_x = \mathrm{A} + \mathrm{B}c^x$, find expressions for l_x and colog p_x, in terms of the constants A, B and c.

Having given the values of the constants A, B and c, set out in detail the steps you would take to construct and check a table of q_x, beginning at age 20.

Examination for Admission to the Class of Fellow (Part III.—Section A).

First Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Given the following data, calculate by the method you consider most suitable the exposed to risk and deaths during the period 1908–1921 (inclusive) in:

(*a*) Select form;

(*b*) Aggregate form;

by independent operations, and show that the latter can be obtained from the former.

State precisely the assumptions you make in the calculation of the ages and durations.

Case	Date of Birth	Date of Entry	Date of Death	Date of Withdrawal	Date of Close of Observations
1	June 1864	Jan. 1905	July 1917	...	...
2	March 1866	Aug. 1906	...	...	31 Dec. 1921
3	July 1866	March 1907	...	June 1910	...
4	Nov. 1867	June 1907	...	June 1920	...
5	Dec. 1865	April 1908	...	...	31 Dec. 1921
6	April 1869	Nov. 1908	Dec. 1916	...	...
7	March 1868	May 1909	...	Feb. 1914	...
8	Jan. 1868	Dec. 1909	...	...	31 Dec. 1921
9	Dec. 1868	July 1910	Sept. 1921	...	...
10	April 1868	Sept. 1910	...	March 1918	...

2. A large assurance company has records, for a number of consecutive years, of its whole life policies in force on 31 December of each year, and of the claims during the year, arranged in the following form:

Year ending 31 *December* 1900 + n. *Age at Entry* x.

Calendar Year of Entry	In force 31 December 1900 + n		Claims Paid during Year 1900 + n	
	No. of Policies	Sum Assured	No. of Policies	Amount Paid
1850				
1				
2				
⋮				
1920				

The ages at entry, x, ranging from 0 to 70, were in all cases ascertained by deducting the calendar year of birth from the calendar year of entry, and it may be assumed that the entries were throughout equally spread over the calendar years.

Describe in detail:

(1) How you would use the data for the purpose of making a yearly comparison between the actual claims and those expected on the basis of the valuation mortality table, and

(2) How you would combine the records for the ten years 1911 to 1920 for the purpose of constructing (a) aggregate and (b) select mortality tables.

In answering (2) you should state whether you would use as your basis the number of policies or the sums assured or both. You should also consider whether it is desirable to divide the data in any way, and indicate what preliminary investigations you would make with the results of (1) in your possession before determining on your course of action.

3. Discuss the relative advantages of "policies", "lives", and "amounts" as a basis for a mortality experience.

State to what extent your general views would be modified in the case of an annuity experience where duplicate contracts for small amounts were frequent (*e.g.*, where there were several instances of 50 contracts taken out at various times by the same annuitant, and where cases of 5 to 10 contracts on the same life were common).

4. What methods have been proposed for graduating an aggregate table by Makeham's formula ?

Give a brief description of the method of application in each case, stating what function would be operated upon for the purpose of graduation, and what special advantages or disadvantages arise.

5. Analyze and develop the summation formula :

$$u'_0 = \frac{[3][3][9]}{81}\{2[5]-2[1]-[7]\}u_0$$

Determine its smoothing coefficient and range.

To what type of data do you consider the formula might usefully be applied ?

Second Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. State precisely the bases upon which the various life tables in connection with the 1911 Census of England and Wales as published in Part I of the Supplement to the 75th Annual Report of the Registrar-General were formed.

Indicate the nature of the rates of mortality under the various tables, as compared with each other, and with earlier Census Tables, and deal fully with the advantages or disadvantages of the method adopted in the selection of the bases of the sectional tables as compared with the bases of the earlier Healthy Life tables.

2. In what way does the exposed to risk of sickness of the Manchester Unity Experience, 1893–1897, differ from that in previous experiences ?

What was the object of the change ?

3. By a short process of interpolation from ungraduated exposed to risk and deaths in an assurance experience the values of p_x have been obtained for quinquennial values of x (15, 20 . . . . . 100) and it is desired rapidly to compare the expectations of life derived from the experience at these quinquennial ages with the corresponding expectations according to a standard table.

Indicate clearly the method you would adopt, giving the formulæ you would employ, and apply the method to find the value of e_{80}, e_{85} and e_{90}, having given

x	$\log p_x$	x	$\log p_x$
75	$\bar{1}$·9669	90	$\bar{1}$·8856
80	·9499	95	·8329
85	·9217	100	·6944

4. Having given the rate of mortality q_x and the force of marriage among bachelors $(bm\mu)_{x+\frac{1}{2}}$, deduce a formula for constructing a combined marriage and mortality table.

5. A life office grants policies under which a return of premiums is made, in respect of total incapacity due to sickness or accident, for each completed month of such incapacity. The return may be claimed for a total period of two years in respect of any one illness or accident, and a claim for the return is treated as a continuation of the last benefit unless twelve months elapse in respect of which no such claim is paid.

The office has done a large amount of this business, extending over a period of 15 years, and desires to ascertain its experience thereunder both for mortality and incapacity.

Draft instructions for the use of your staff in deducing such combined experience.

6. Describe generally the method of osculatory interpolation and demonstrate the fundamental formula when the problem is to provide a smooth juncture at u_0 and the data given are u_{-2}, u_{-1}, u_0, u_1 and u_2.

What are the advantages or disadvantages of the system for the purpose of graduation?

Examination for Admission to the Class of Fellow (Part III.—Section B.)

First Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

The total business (all "with profit" whole-life) on the books of an office as at 31 December 1920, is given in the following table:

Office Year of Birth	Sum Assured	Rever-sionary Bonus	Annual Office Premium	Reduction of Premium by Bonus	Reduction of Premium by Purchase	Net H^M 3 % Premium corresponding to full Office Premium
1865	10,000	1,500	300	...	10	230
1864	8,000	1,000	250	15	...	190
1863	7,500	800	200	...	...	155
1862	6,000	450	190	...	20	145
1861	5,000	...	160	50	...	125
	36,500	3,750	1,100	65	30	845

New business completed during the year 1921:

Age next Birthday	Effected	
56	July	Sum Assured £1,000 at an office annual premium of £58.
57	May	Sum Assured £1,500 for a first premium of £145 and future annual premium of £85. (Equivalent tabular office annual premium £90).
58	August	Sum Assured £100 at a single premium of £74. (Equivalent tabular office annual premium £6 6*s*.).

Cancelments and alterations during the year 1921:

Office Year of Birth	Sum Assured	Reversionary Bonus	Intermediate Bonus	Office Annual Premium	Net Premium	Remarks
1865	200	50	3	4	2·8	Death after payment of premium.
1862	500	60	7·5	12	9	Death after payment of premium.
1865	500	...	...	15	11·7	Surrendered for £135 before payment of premium.
1862	500	...	...	15	11·7	Converted to a paid-up policy for £220 with profits, before payment of premium.
1864	...	150	...	...	...	Reversionary bonus taken in a cash payment of £75.
1865	...	100	...	...	...	Reversionary bonus used to reduce annual premiums by £5 commencing 1921.

Make a valuation of the business as at 31 December 1921, by the H^{M} 3 per-cent Table.

Net premiums are calculated at age next birthday at entry. The office year of birth deducted from the calendar year of valuation = valuation age = age next birthday at entry + nearest duration.

You may assume equal distribution of premium income.

Having given the following items in the revenue account:

Fund at beginning of year ...	£16,890
Interest, less tax	680
Profit on investments realized. .	50
Commission	80
Expenses of management ...	140

Complete the account and ascertain the surplus.

Analyze the surplus as completely as possible from the information at your disposal and state with reasons the rate of compound reversionary bonus you would advise the directors to declare.

Second Paper.

1. Describe the principal methods of grouping whole-life assurances for valuation according to the assumptions made as to age at entry, duration and valuation age. State generally the comparative effects of each.

Which method do you prefer, and why?

2. A life office in the past has declared the same rate of compound bonus to whole-life and endowment assurance policies. The directors have raised the question as to whether apart from depreciation the endowment assurance class has provided its share of the surplus.

Explain carefully how you would make the investigation.

3. Previous to the war a life office valued on a $2\frac{1}{2}$ per-cent net premium basis and maintained a 30*s*. compound bonus. At a quinquennial valuation during the war the valuation rate of interest was increased to 3 per-cent and a bonus was declared at a considerably reduced rate. At the succeeding valuation after the war the 3 per-cent valuation was maintained, but owing to further depreciation in securities no bonus was declared. The whole of the depreciation was written off on each occasion.

If at the next valuation the normal profits of the quinquennium are sufficient to declare a 30*s*. compound bonus, while in addition there is a large appreciation in securities, how would you deal with this profit from appreciation.

Discuss the question from all points of view.

If it is decided to divide some or all of the profit from this source, give your views as to the most equitable method of doing so.

4. How would you investigate the question as to the minimum extra loading reserve required under whole-life limited premium policies?

5. State how at a quinquennial investigation you would value:

(1) Convertible term policies both before and after conversion.

(2) Deferred assurances for children vesting at age 21 where on death of parent premiums cease up to age 21.

(3) Whole-life policies, non-profit for 10 years but thereafter with profits, under which a level annual premium is paid.

(4) Deferred annuities without return of premium.

6. Set out concisely the information required under Schedule IV of the Assurance Companies Act, 1909, in respect of life assurance business.

Examination for Admission to the Class of Fellow (Part IV.—Section A).

First Paper.

"A Short Collection of Actuarial Tables" and the Supplement will be supplied for use in answering this Paper.

1. State by what Act or Acts of Parliament life insurance companies are now governed. Give a brief account, with particulars, regarding the various Schedules now required from life insurance companies, and say how often they have to be furnished. What deposits are necessary from all classes of insurance companies and when have they to be made?

2. Discuss the bases upon which you would calculate the surrender values of children's deferred assurances before and after the attainment of age 21.

3. A life office with a large number of investments in absolute and contingent reversions and reversionary life interests needs very promptly to determine whether the mortality experienced during a certain year has as a whole been greater or less than anticipated by the valuation made each year.

How would you proceed to investigate the matter?

4. A is entitled to the absolute reversion on the death of the present life tenant, a lady aged 60, to one-half of the following fund:

£8,000 National War Bonds (1928);

Freehold ground rents in the City of London amounting to £1,000 per annum with reversion to rack rents, estimated at £3,500, in 1935.

Leasehold house with lease having 45 years to run, at present let at a rent of £200, on a pre-war lease, expiring in 1930.

You are asked to advise on the selling price of A's reversion in the open market.

5. Upon the death of A, without issue, B will, provided he be then living and retain the family name and arms, inherit an income of £8,000 a year from settled estates.

A is 75 with a wife but no child and the issue risk can be covered at a single premium of £3 3*s*. per-cent.

B's life has just been accepted at age 35 at the tabular annual premium of £2 5*s*. per-cent for a whole life non-profit assurance, and the "name and arms" risk can be covered at a single premium of £2 2*s*. per-cent.

Upon what terms could an advance of £5,000 be made and for what amount should the various policies be effected? It is desired to realize 7 per-cent (gross) interest in reversion and 5 per-cent free of tax when in possession.

Second Paper.

"A Short Collection of Actuarial Tables" and the Supplement will be supplied for use in answering this Paper.

1. A life office, assessed for income tax purposes on its interest income, had for some years before the war been a large purchaser of reversions. Draft a report to the directors dealing with the desirability or otherwise of a change of policy in the altered circumstances produced by the war.

2. X, an only son aged 30, will become entitled, if living at the death of his mother, aged 60, who is of unsound mind, and has made no will, to the following fund in Court:

£15,000 Irish Land Guaranteed $2\frac{3}{4}$ per-cent Stock, and
£6,350 Bank of Ireland Stock (present dividend 12 per-cent).

What is the maximum advance that should be made for a deferred charge upon X's expectancy and what policies are required? It may be assumed that the life of X can be assured against the life of his mother at an annual premium of £1 8*s*. 0*d*. per-cent, and that a satisfactory charge can be obtained.

3. The following rent charges are well secured upon a large landed estate, namely:

£200 p.a. to A	(born July 1858)		but all are charged with an annuity of £250 per annum to D (born December 1847).
£300 ,, B	(,, Feb. 1863)		
£400 ,, C	(,, Oct. 1891)		

All the lives are female and the annuities are payable subject to income tax, D's annuity being distributed proportionately amongst the others. The combined interests of A, B and C are for sale and all three are willing to be medically examined. If accepted at ordinary rates (use Carlisle net 3 per-cent premiums) what offer could be made for purchase by a life office to yield 5 per-cent (free of tax)?

4. A, aged 65, his brother, B, aged 60, and A's son, C, aged 30, are interested in the following fund:

£10,000 Funding Loan,
£3,000 Bank of England Stock (present dividend 10 per-cent).

A and B have successive life interests in the fund and C is entitled to the reversion to the fund on the death of the survivor of A and B provided he be then living. Otherwise the fund goes to D on the death of the survivor of A and B. Apportion the fund between A, B, C and D, upon the assumption that all the lives are in good health and are anxious for the fund to be divided.

5. A, aged 45, has an absolute reversion on the death of his uncle, B, aged 78, to the following fund :

£15,000 3½% Conversion Loan (red. after 1961) ;

£11,000 Straits Settlements 6% Stock (1936/51) ;

£18,000 Midland Railway 2½% Debenture Stock ; and

£218 per ann. Scinde, Punjaub and Delhi "B" Annuity (1958).

A wishes to obtain a present income but prefers not to sell the whole of his reversion, as he regards B as a bad life. How would you advise A to proceed under the circumstances ? Indicate the terms which you would regard as appropriate.

Examination for Admission to the Class of Fellow (Part IV.—Section B).

First Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. Define and distinguish between (1) Declaration, (2) Warranty, (3) Representation.

"Whether a statement is material or no is a matter of fact." What bearing has this upon statements made, or omitted to be made, in a proposal for life assurance ? Give particulars of any matters which in this connection have been judicially considered to be material and requiring disclosure.

2. The death of the life assured under an own life policy has been proved.

What evidence of title will be required and whose discharge for the policy monies will be necessary in the following circumstances :

(*a*) If the policy has never been assigned and :

1. The assured leaves a will,
2. The assured dies intestate,
3. The assured is bankrupt ;

(*b*) If the policy has been assigned:
1. Absolutely,
2. By way of mortgage; and

(*c*) If, without the execution of a formal assignment the policy has been deposited with a third party and notice of charge has been furnished to the company.

What variations, if any, will be required in the documents of title if the assured dies domiciled outside the United Kingdom?

3. How does bankruptcy affect life assurance policies? Is it safe for a company to pay cash surrender values without searching the files to ascertain if the policyholder has been adjudicated bankrupt. Give reasons, mentioning particularly any recent decisions or opinions of which you may be aware.

4. The Stamp Act, 1891, requires policies of assurance to be stamped and imposes a penalty on an insurer neglecting to issue such a policy. What is the time limit and the amount of the penalty?

A company issues a 30-year endowment assurance policy for £100, under which a guaranteed bonus of £4 is added in respect of each premium paid. What amount of stamp will be required?

What special clause must be inserted in an annuity bond when the purchase money does not exceed £500? Give the reason.

What are the obligations of a life office in regard to stamps upon deeds of assignment relating to a policy when making a payment thereunder?

5. An endowment assurance policy to mature on the anniversary of the policy preceding the attainment of age 60 was issued five years ago on a life then stated to be 35 years next birthday. Evidence is now produced showing that the age should have been given as 38 years next birthday. Draft a letter to the policyholder stating the means by which the error may be rectified, offering alternative methods if you consider it desirable, and set forth the appropriate form or forms of endorsement by which you would give effect to the alteration.

6. Draft a report to your directors on the desirability or otherwise of transacting group assurance business.

Deal with the matter fully, describing the various types of policy that might be issued.

7. The following are *extracts* from the accounts of a proprietary life assurance company:

	31st Dec. 1920	31st Dec. 1921
Revenue Account.	£	£
Fund at beginning of year	2,000,000	1,980,000
,, end of year	1,980,000	2,130,000
Premiums	172,000	175,000
Interest, gross	97,000	108,000
Less tax	21,000	19,000
Interest, net	76,000	89,000
Claims by death	91,000	75,000
,, maturity	13,000	17,000
Surrenders	11,000	10,000
Expenses and Commission	27,000	29,000
Written off investments	65,000	1,000
Investments realized	(Loss) 51,000	(Profit) 28,000
New business issued.		
Number of policies	667	400
Sum assured	£500,000	£360,000
Renewal premiums	£17,500	£12,300
Balance Sheet.	£	£
Mortgages	380,000	450,000
Loans on Policies	120,000	150,000
British Government Securities	780,000	600,000
Municipal and County Securities, U.K. ...	100,000	120,000
Debentures	30,000	80,000
Outstanding premiums	9,000	12,000
Deposits for fixed terms...	40,000	200,000

Draft in outline the points to which you would draw the chairman's attention to enable him to deal fully with the transactions of the year 1921 at the Annual Meeting, and suggest the lines upon which the report may be made.

Second Paper.

"A Short Collection of Actuarial Tables" will be supplied for use in answering this Paper.

1. You are asked by a new company to draw up tables for publication in its first prospectus. The company contemplates developing its business by appointing as spare-time agents men engaged as clerks in large business houses.

What bases would you employ for the calculation of the rates of premium for with and without profit policies under (1) whole life assurances, whole term; (2) whole life assurances with premiums ceasing at ages 55 and 60; (3) endowment assurances maturing at ages 55 and 60?

Give the formulæ you would use, stating clearly the Tables of Mortality and rates of interest involved and the type of bonus you contemplate giving to the with-profit policyholders.

2. How would you deal with the following enquiries for policies on a life aged 40 next birthday:

Twenty and twenty-five-year term endowment assurances with profits with a reduced premium for the first ten years.

Whole life policy with profits by twenty annual premiums the sum payable in event of death within the first ten years to be restricted to a return of the amount paid in premiums.

3. A medical practitioner aged 35 years next birthday asks for the quotation of the single premium for an assurance of £1,000 on his life to become payable only in the event of his death occurring between the ages of 55 and 65, no benefit whatever to be granted, or return of premium to be made if death should occur at any other time.

What enquiries would you make before furnishing a quotation and if you were satisfied as to those enquiries, what rate would you quote, subject to medical examination and the usual power of the directors to decline the proposal?

4. How would you calculate:

(*a*) the annual premium; (*b*) the single premium for the option to effect an ordinary whole life policy ten years hence at the ordinary rate of premium for the then age?

5. Draft a report on the desirability of issuing policies providing for waiver of premiums on disability, indicating the nature of the conditions or limitations to be imposed and the basis you would adopt in arriving at the additional premiums to be charged for the privilege.

6. Define the term "first class life", as applied to a life proposed for assurance. By what means can an office protect itself against the issue of policies at first class rates on under average lives?

To what extent do you consider it necessary or desirable to obtain particulars of family history? State in outline the manner in which you would give effect to family history when considering a proposal for a whole life assurance.

7. A person aged 30 next birthday is insurable under a whole life policy at ordinary rates subject to payment for ten years of a climatic extra premium of one per-cent per annum. On what terms would you recommend acceptance under a 20-year endowment assurance policy—the ordinary tabular rate of premium to be paid but the sum assured to be reduced in the event of death occurring whilst the assured is abroad?

8. What are your views with regard to granting annuities at specially favourable rates on lives in impaired health?

What investigations would you institute before considering the grant of special terms?

Examination for Admission to the Class of Fellow (Part IV.—Section C).

First Paper.

1. A group of shipping companies provides the following pensions and gratuities to officers and engineers in their service :

(*a*) *Pensions.*—Subject to completion of 25 years' service, on the attainment of age 52 and before the attainment of age 57, a master, officer or engineer may retire and receive a pension in accordance with the following scale :

Retiring Age	Pension per Annum		
	Master	Chief Engineer	Chief Officer or Second Engineer
	£	£	£
52	188	150	75
53	194	155	80
54	204	163	85
55	216	172	90
56	232	185	95

On the attainment of age 57 and subject to completion of 30 years' service, a master, officer or engineer must retire and receive a pension as follows :

Rank on Retirement	Pension per Annum
	£
Master	250
Chief Engineer	200
Chief Officer or Second Engineer	100

Masters and chief engineers must have completed five years' service in their respective ranks in order to qualify for pension on the higher scales, otherwise they receive a pension on the lowest scale.

(*b*) *Gratuities on retirement through incapacity.*—In the event of a master, officer or engineer being permanently incapacitated for further service before qualifying for a pension he shall receive a gratuity equivalent to one month's pay at the time of retirement for each 12 months' service completed.

Note.—Officers and engineers retiring below the rank of chief officer or second engineer are not entitled to any pension or gratuity.

No contributions are paid by the staff, and the companies who have hitherto paid benefits out of revenue now desire to fund

their liabilities by means of a capital payment in respect of the accrued service of their existing staff and an annual contribution consisting of a flat percentage of the total pay roll of all ranks of officers and engineers in respect of future service of existing and future staffs.

You may assume that officers are recruited direct to officer class at a uniform age at entry.

The total numbers involved are about 1,500 and the staffs may be assumed to have reached a stationary condition.

State precisely how you would proceed to make the necessary estimates.

2. Information has been obtained as to the size of the families from which the patients in a certain sanatorium have been drawn. The figures are as follows :

Size of family...	1	2	3	4	5	6	7	8	9	10
Number of cases	8	15	20	20	30	30	20	15	12	10
Total 180										

The average family is thus 5·4.

From this result it is suggested that tuberculous families are on the average larger than those of normal stocks.

Discuss the limits within which such an inference is permissible.

3. Describe carefully the method by which the proportion of persons commencing sickness in any year of age can be deduced from the Manchester Unity Tables.

The experience of a large friendly society is available in respect of the following sickness periods :

"first 13 weeks", "second 13 weeks", "second 26 weeks", "second 12 months", "after 2 years."

You are required to find the amount of sickness falling within the six weeks of claim following the first 3 days. Show how you would proceed.

Second Paper.

1. An old established railway provident society consisting of over 50,000 members grants the following benefits :

Disablement allowance :

12*s.* per week in case of temporary disablement from work for a period of 52 weeks.

Disablement is regarded as continuous unless a period of 12 months intervenes between any two claims.

Funeral benefits :

Death of member, £10.

Death of member's wife, £5.

If the wife's benefit is paid after retirement then the member's benefit is reduced by £5.

Retiring gratuity upon disqualification for duty :

After 5 but less than 10 years' membership,				£12 10*s*.
,, 10 ,,	,, 15 ,,		,,	£25.
,, 15 ,,	,, 20 ,,		,,	£37 10*s*.
,, 20 years' membership	...	...		£50.

Pension on retirement from service at age 65 or after age 60 if disqualified for duty ... 10*s*. per week.

Pension after 20 *years' membership*, if disqualified for duty before age 60 ... ... 5*s*. per week.

Disablement allowance ceases after 52 weeks, and membership ceases unless member resumes duty for six months or is entitled to a pension.

The members pay weekly contributions, varying in amount with the age at admission, which cease on leaving the service of the company.

The company contributes an annual sum of £25,000 per annum and 1*d*. per week in respect of each member in excess of 50,000.

You are requested to take out the experience of the society and to make the valuation. State fully upon what principles you would proceed, and why.

2. A company with over 10,000 employees desires to establish a widows' and orphans' fund in conjunction with an existing pension fund.

It is proposed that the widow of a pensioner or of a member dying in the service of the company shall, subject to 10 years' membership, be entitled to an annuity of 1 per-cent of the total salary on which contributions have been or are assumed to have been paid to the new fund.

Orphans under 15 (but not more than 3 at any one time—if more than 3 then the 3 youngest), to be entitled to an annuity per child of one-fourth the widow's annuity—payment to continue until age 15 or earlier death. An orphan's annuity ceasing by death may be transferred to any other child under age 15 not in receipt of an annuity.

Maximum family annuity £250 per annum. Minimum widow's pension £30 per annum.

Any member leaving the service of the company or retiring with no wife living or children under 15, or any pensioner whose wife predeceases him leaving no children under 15, shall be refunded the whole of his own contributions.

On a member dying before completion of 10 years' membership or, after that period, without leaving a widow or children under 15, his representatives are to be entitled to a return of his own contributions.

The contributions are to be a flat-rate percentage of salary and to be divided equally between the company and the members. Contributions to cease on death, secession or retirement.

Membership is to be optional for all existing employees over age 25 but compulsory for all future employees upon attainment of that age.

Existing employees over age 25 are to be credited with membership back to age 25 and the cost is to be met by a donation from the company.

The whole of the data and statistics of the pension fund including existing and past salaries are available, and in addition, the family and marital statistics of the whole staff.

State how you would proceed to ascertain the rate of contribution and the amount of the company's donation.

PROCEEDINGS OF THE INSTITUTE.—Session 1921-1922.

First Ordinary Meeting, 31 *October* 1921.

The President (Sir Alfred Watson, K.C.B.) in the Chair.

A paper entitled "An investigation into the mortality experienced by life tenants under reversions, with some conclusions drawn therefrom" was submitted by the Author, Mr. E. H. Lever.

The following gentlemen took part in the discussion:—Sir Joseph Burn, K.B.E., Messrs. C. Carpmael, W. Mouat Jones, A. D. Besant, C. R. V. Coutts, W. P. Phelps, E. W. Phillips, W. R. Moore, F. L. Collins, and the President.

Second Ordinary Meeting, 28 *November* 1921.

The President (Sir Alfred Watson, K.C.B.) in the Chair.

Mr. Hugh McCall Ferguson, F.F.A., was elected an Associate of the Institute.

A paper entitled "On the relation between the course of wholesale prices of commodities and the market value of various classes of securities" was submitted by the Author, Mr. S. J. Perry.

The following gentlemen took part in the discussion:—Messrs. A. J. C. Edwards, Hartley Withers (a visitor), Geoffrey Marks, A. D. Besant, S. G. Warner, H. E. Raynes, O. T. Falk, and the President.

Third Ordinary Meeting, 30 *January* 1922.

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

A paper entitled "A short method of constructing select mortality tables. Further developments" was submitted by the Author, Mr. George King.

The following gentlemen took part in the discussion:—Messrs. A. G. Paton, A. Henry, R. D. Anderson, C. W. Kenchington, R. E. Underwood, D. C. Fraser, and the President.

Fourth Ordinary Meeting, 27 *February* 1922.

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

A paper entitled "Results of an investigation into the effect of different valuation bases on surplus" was submitted by the Author, Mr. C. H. Maltby.

The following gentlemen took part in the discussion:—Messrs. S. J. Rowland, R. C. Simmonds, A. E. Brown, A. S. Holness, A. Henry, W. P. Elderton, and the President.

Fifth Ordinary Meeting, 27 *March* 1922.

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

A paper entitled "Austrian national life tables" was submitted by the Author, Mr. G. W. Richmond.

The following gentlemen took part in the discussion:—Messrs. V. P. A. Derrick, F. A. A. Menzler, W. P. Elderton, A. Henry, F. B. Wyatt, S. J. H. W. Allin, and the President.

Sixth Ordinary Meeting, 1 *May* 1922.

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

A paper entitled "On the valuation of endowment assurances by select tables", by Mr. E. H. Brown, was submitted.

The following gentlemen took part in the discussion:—Messrs. H. H. Tayler, A. Levine, R. E. Underwood, G. W. Richmond, H. J. P. Oakley, and the President.

The Seventy-fifth Annual General Meeting, 12 *June* 1922.

The President (Sir ALFRED WATSON, K.C.B.) in the Chair.

The proceedings at the Annual General Meeting will be found on page 355.

REPORT, 1921–1922.

The Council have the pleasure to report to the Members upon the work of the Institute during the Session of 1921–1922, the seventy-fourth year of its existence.

There has been a *decrease* of 28 in the total number of members, as compared with the previous year. At the end of the official year in which the Institute was incorporated by Royal Charter the number of Members was 434; twenty-six years later, at 31 March 1911, it was 934. Since that time the numbers have been as follows:

On 31 March	Fellows	Associates	Students	Corresponding Members	Total
1912	278	354	268	20	920
1913	282	355	252	19	908
1914	295	358	238	19	910
1915	304	361	263	17	945
1916	308	345	247	17	917
1917	303	344	231	18	896
1918	295	332	215	18	860
1919	288	330	205	18	841
1920	305	345	197	18	865
1921	324	345	201	18	888
1922	319	344	179	18	860

The following schedule shows the additions to, and the changes and losses in the membership which have occurred during the year ending 31 March last:

Schedule of Membership, 31 *March* 1922.

	Fellows	Associates	Students	Corresponding Members	Total
i. Number of Members in each class on 31 March 1921	324	345	201	18	888
ii. Withdrawals by					
(1) Death	6	5	2	...	67
(2) Resignation or otherwise	7	17	30	...	
	311	323	169	18	821
iii. Additions to Membership					
(1) By Election	...	2	...	...	39
(2) By Examination	...	...	35	...	
(3) By Re-instatement	...	1	1	...	
	311	326	205	18	860
iv. Transfers					
(1) By Examination: *from* Associates	...	7	...	...	...
to Fellows	7	...	...	...	...
	318	319	205	18	860
(2) By Examination: *from* Students	...	...	1	...	...
to Fellows	1	...	...	...	..
	319	319	204	18	860
(3) By Examination: *from* Students	...	...	25	...	...
to Associates	...	25	...	...	...
v. Number of Members in each class on 31 March 1922	319	344	179	18	860

There are also 227 candidates admitted as Probationers, and 79 as Students conditionally on their passing Part I of the Examination. These

are not included in the above Schedule of Membership. The number in these two classes at 31 March 1916 was 172 and 73. Since that date the numbers have been as follows :

On 31 March	Probationers	Conditional Students	On 31 March	Probationers	Conditional Students
1917	173	67	1920	230	79
1918	156	70	1921	221	110
1919	169	73	1922	227	79

The Council have, with great regret, to report the loss by death, since the last Annual Meeting, of six Fellows, Messrs. W. Browne, J. C. Denmead, Gordon Douglas, J. C. Peter, W. C. Sharman, and R. A. C. Thomas ; five Associates, Messrs. R. W. Barton, W. G. FitzGerald, W. M. Monilaws, D. F. Park, and G. Watt; and two Students, Messrs. T. Arnold and W. J. Slimon.

Mr. Gordon Douglas, who was a Past-President of the Faculty of Actuaries in Scotland, had served on the Council of the Institute. As Honorary Secretary of the Scottish Section he took a prominent part in the work of the Joint Committee of the Institute and the Faculty which had charge of the investigation of the mortality experienced by British Life Offices during the period 1863–93.

Mr. W. C. Sharman, one of the younger Fellows, had also served on the Council, and had been a member of the Board of Examiners. Mr. Sharman rendered valuable services to the profession as the writer, for some time, of the Legal Notes for the *Journal*.

A bronze tablet has been placed in the Hall in memory of the men of the Institute whose lives were given in the service of their Country during the War 1914–1919. The total expenditure in connection with the Memorial was £376. 11*s*. 0*d*., all of which, with the exception of a small amount received for deposit interest, was contributed by Members and Probationers of the Institute. A report of the proceedings at the unveiling ceremony will appear in the next issue of the *Journal*.

The Annual Subscriptions and the Entrance Fees appearing in the Revenue Account amounted to £2.102. 12*s*. 6*d*., as compared with £2,204. 9*s*. 6*d*. in the previous year. The Income and Expenditure for the year were £2,372. 14*s*. 1*d*. and £2,666. 17*s*. 2*d*. respectively.

The Messenger Legacy Fund and the Brown Prize Fund have been amalgamated. The combined Fund and the G. F. Hardy Memorial Fund have had separate securities allocated to them, and now appear in the Statement of Account separately from the general funds of the Institute. This arrangement has been adopted in connection with the scheme of prizes for sessional papers and other works of exceptional merit. Under this scheme the income of the combined Fund will be utilized in the provision of prizes to be granted annually after consideration by a special Committee of the papers submitted to the Institute and the contributions to the *Journal* during the previous twelve months : all members of the Institute will be eligible for these prizes except members and ex-members of the Council. The income of the G. F. Hardy Memorial Fund will be applied primarily to grants in cases where expense is incurred, or expected to be incurred, in connection with research work. An application for a grant from this Fund may be made privately to the Honorary Secretaries at the beginning of the

[*Continued on page* 352.]

Dr. *Revenue Account for the*

1921. £ s. d.		£ s. d.	1922. £ s. d.
958 13 0	Subscriptions—Fellows ...	964 19 0	
700 7 0	Associates	699 6 0	
204 4 6	Students	195 16 6	
142 16 0	Probationers	123 18 0	
2,006 0 6		1,983 19 6	
77 14 0	One Compounded for	25 4 0	
2,083 14 6			2,009 3 6
14 14 0	Entrance Fees—Associates	4 4 0	
62 9 6	Students	43 1 0	
43 11 6	Probationers	46 4 0	
120 15 0			93 9 0
467 5 6	Dividends and Interest, *less* tax...		270 1 7
865 8 2	Balance ...		294 3 1
£3,537 3 2			£2,666 17 2

Publications Account for the

£ s. d.		£ s. d.
247 0 2	Sales of *Journal* ...	189 10 11
376 13 3	Sales of other Publications ...	348 15 5
70 3 7	Stock in hand, *less* Depreciation...	1 0 0
96 6 6	Balance ...	375 13 10
£790 3 6		£915 0 2

Balance Sheet,

£ s. d.	LIABILITIES.	£ s. d.	£ s. d.
8,004 17 0	Balance at 1 April 1921 ...	7,139 8 10	
865 8 2	Balance of Revenue Account ...	294 3 1	
7,139 8 10			6,845 5 9
50 0 5	Sundry Unpaid Accounts ...		25 1 8
13 2 6	Examination Fees, 1922 ...		118 2 6
...			£6,988 9 11

Trust Funds. Income and Expenditure Account

	G. F. Hardy Memorial Fund. £ s. d.	Messenger and Brown Prize Fund. £ s. d.	Total. £ s. d.
Interest, *less* tax ...	32 16 8	33 13 8	66 10 4
Tax Refunded ...	14 6 3	11 13 6	25 19 9
	£47 2 11	£45 7 2	£92 10 1

Trust Funds. Balance Sheet,

	G. F. Hardy Memorial Fund. £ s. d.	Messenger and Brown Prize Fund. £ s. d.	Total. £ s. d.
Capital Account at 1 April 1921 ...	959 11 4	887 0 0	1,846 11 4
Balance of Income Account ...	7 2 11	19 2 2	26 5 1
	£966 14 3	£906 2 2	£1,872 16 5

year ending 31 *March* 1922. **Cr.**

1921. £ s. d.		£ s. d.	1922. £ s. d.
53 16 8	Library—Binding, Purchases, &c.	...	43 17 6
66 13 2	Meetings	...	70 2 6
...	Legal Charges	...	8 2 0
761 19 11	Examination Charges	421 7 2	
356 9 6	*Less* Fees received from Candidates	197 8 0	
405 10 5			223 19 2
707 2 0	Tutors for Classes in Parts I and II	635 5 0	
521 17 0	*Less* Fees received from Students	446 5 0	
185 5 0			189 0 0
600 0 0	Office Expenditure—Rent	600 0 0	
669 7 5	Salaries	675 8 6	
96 3 5	House Expenses	90 16 9	
42 4 10	Fire and other Insurance	41 12 7	
194 0 3	Stationery and Printing	267 3 10	
59 4 2	Postage, Telegrams and Telephone	72 17 7	
15 17 8	Sundries	8 2 11	
1,676 17 9			1,756 2 2
19 6 4	*Lectures on Finance and Statistics*		...
26 19 6	*Valuations for the Royal Patriotic Fund Corporation (out of pocket expenses)* ...		...
1,006 7 10	*Loss on Sale of Stock Exchange Securities*		...
96 6 6	Balance of Publications Account		375 13 10
£3,537 3 2			£2,666 17 2

year ending 31 *March* 1922.

£ s. d.		£ s. d.
144 12 0	Stock at 1 April 1921	70 3 7
523 18 8	Cost of *Journal*, Nos. 275, 276	744 9 1
46 0 0	*Cost of Supplement to "Short Collection of Actuarial Tables"*	...
75 12 10	Binding, Advertising, &c.	100 7 6
£790 3 6		£915 0 2

31 *March* 1922.

£ s. d.	ASSETS.	£ s. d.
...	£2,534 0*s.* 11*d.* 3½ per-cent Conversion Stock	1,488 14 11
...	£3,471 14*s.* 4*d.* 4 per-cent Funding Stock, 1960–90	2,482 10 0
...	£4,700 0*s.* 0*d.* Local Loans 3 per-cent Stock	2,502 19 0
70 3 7	Stock of Publications in hand, *less* depreciation	1 0 0
683 6 8	Cash at Bank and in hand	381 16 9
107 2 11	Interest accrued but not received, *less* tax	131 9 3
...		£6,988 9 11

for the year ending 31 *March* 1922.

	G. F. Hardy Memorial Fund. £ s. d.	Messenger and Brown Prize Fund. £ s. d.	Total. £ s. d.
Grants for Investigation or Research Work	40 0 0	...	40 0 0
Prize Awarded	...	26 5 0	26 5 0
Balance	7 2 11	19 2 2	26 5 1
	£47 2 11	£45 7 2	£92 10 1

31 *March* 1922.

	G. F. Hardy Memorial Fund. £ s. d.	Messenger and Brown Prize Fund. £ s. d.	Total. £ s. d.
£1,603 8*s.* 8*d.* 3½ per-cent Conversion Stock, 1960–90...	959 11 4	...	959 11 4
£1,456 14*s.* 8*d.* 3½ per-cent Conversion Stock, 1960–90...	...	887 0 0	887 0 0
Cash at Bank	7 2 11	19 2 2	26 5 1
	£966 14 3	£906 2 2	£1,872 16 5

We have examined the foregoing Accounts with the books and vouchers of the Institute and certify them to be correct.

D. M. CARMENT,
ALEX. S. SELLAR, } *Auditors.*
E. A. WOODALL,

5 *May* 1922.

work, at any time during its progress, or on its completion. Such applications will be dealt with by the Officers of the Institute, and the fact that a research grant has been made in any particular case will not render the recipient ineligible for an award from the Prize Fund in respect of the completed work.

On the first occasion of putting the scheme into operation a prize of twenty-five guineas was awarded to Mr. G. S. W. Epps for his paper on Superannuation Funds, and two grants, each of £20, were made for investigation work which resulted in the contribution of two papers to the Proceedings of the Institute during the Sessions 1920–21 and 1921–22.

For the Examinations held in the United Kingdom and at the usual centres Over-seas on the 24, 25 and 26 April last, 445 entries were received, namely:

162	for Part	I, Sec. A.
107	,,	I, Sec. B.
61	,,	II.
46	,,	III, Sec. A.
34	,,	III, Sec. B.
19	,,	IV, Sec. A.
9	,,	IV, Sec. B.
7	,,	IV, Sec. C.

The results will be duly announced.* The Council warmly acknowledge the valuable services of the Board of Examiners, and also those of the Honorary Supervisors.

Revised Regulations and Syllabus of the Examinations, to come into force in 1923 as regards Parts I and II and in 1924 as regards Parts III and IV, have been issued.

The *Journal* will be published three times a year until further notice, the Number for March last being the first of the three Parts which will comprise the volume for 1922. The arrangement with Mr. R. A. Bateman, B.Sc. (Econ.), Barrister-at-Law, to contribute the Legal Notes to the *Journal*, has been renewed.

The *Text-Books* on Mathematics and Life Contingencies, by Messrs. A. Henry and E. F. Spurgeon respectively, will shortly be in the printers' hands. It is hoped that the volumes will be published before the opening of the next session.

The work in connection with the investigation of the mortality experience of British life annuitants during the years 1900–20 is proceeding, and arrangements have been made for a continuous investigation of annuitant mortality experience from 1921.

A letter has been received from The Lords Commissioners of His Majesty's Treasury, thanking the Council for assistance in connection with the valuation of War Pensions for the Reparations Commission.

9 *May* 1922.

* See pp. 353–4.

EXAMINATIONS, APRIL 1922.

Examinations were held on the 24th, 25th and 26th April 1922, at London, Liverpool, Birmingham, Cardiff, Norwich, Edinburgh, Dublin, Melbourne, Sydney, Brisbane, Wellington, Montreal, Toronto, Ottawa, Winnipeg, Bombay, Calcutta, and Singapore, with the following results :

Part I.—Section A.

One hundred and sixty-two candidates sent in their names, of whom one hundred and forty-seven presented themselves, and nineteen passed, namely :

Bowrey, Miss M. L.	Irwin, J. O.	Strachan, W. B.
Butt, L. M.	Layton, E. C.	Tanner, E. L.
Geary, R. C.	Llewellyn, G. F.	Trustam, C. F.
Ghatak, R.	Offord, L. W.	Wood, C. F.
Gunlake, J. H.	Pelling, W. E.	Wood, E. G.
Haines, P. H.	Rao, N. R. Jaya	
Henderson, J.	Stern, C. M.	

Part I.—Section B.

One hundred and seven candidates sent in their names, of whom ninety-three presented themselves, and twenty-seven passed, namely :

Butler, A. G.	Haslehurst, G.	Pelham, E. N.
Callum, J. A.	Heathershaw, P. C.	Pelling, W. E.
Cunningham, P. G.	Hewens, L. H.	Pollock, W.
Dinnage, R.	Janes, J. E.	Rao, N. R. Jaya
Fagg, R. J.	McLaren, W. S.	Snelling, B. S.
Gates, W. L.	Mankey, W. S.	Tharp, H. W.
Goffin, W. J.	Moss, G. F.	Trustam, C. F.
Gough, R. A.	Nickerk, J. van	Turnbull, J.
Grealy, L. J.	Norton, F. R.	Wenborn, N. S.

Part II.

Sixty-one candidates sent in their names, of whom fifty-seven presented themselves, and twenty-one passed, namely :

Bourke, G. W.	Gill, D. H.	Lockwood, B.
Bray, J. F. L.	Hickox, W. E. H.	Lynch, L.
Briscoe, H. J.	Holmes, G. L.	Mann, A. H.
Brown, F. S.	Joshi, K. Y.	Pearson, A. M.
Dicken, H.	Kingham, C. E.	Shine, J. N.
Elrick, W.	Lavery, R. W.	Stark, A. W.
Gibberd, J. A.	Legge, Miss P. C.	Strachan, W. B.

Part III.—Section A.

Forty-six candidates sent in their names, of whom thirty-nine presented themselves, and twenty-one passed, namely:

Carpenter, T. B. B.	Garland, W. E.	Pedoe, A.
Davie, J.	Goodfellow, P.	Sanger, C. W.
Davis, Miss D. B.	Gregory, Miss G. C.	Shaw, E. H.
Dodwell, Miss M. E.	Hornsby, S. J.	Starke, L. G. K.
Eames, G. S.	Keady, P. J.	Taylor, J. A. G.
Finch, G.	Milnes, H. L.	Thomas, J, H.
Gardner, W. F.	Mitchell, H. H.	Usherwood, K. A.

Part III.—Section B.

Thirty-four candidates sent in their names, of whom thirty-three presented themselves, and eleven passed, namely:

Clough, W. H.	Loraine, W. E. P.	Scott, Miss H. M.
Cope, A.	Mitchell, H. H.	Starke, L. G. K.
Creese, H. R.	Murrell, R.	Tutill, H. L.
Gregory, Miss G. C.	Perks, W.	

Part IV.—Section A.

Nineteen candidates sent in their names, all of whom presented themselves, and fourteen passed, namely:

†Davis, C. M.	†Reynolds, P. C.	†Watson, A. D.
†Gostelow, C.	†Scott-Wilson, A. B.	†Welsh, W.
†Morton, F. W.	†Shrewsbury, A. H.	†Wolfenden, H. H.
†O'Brien, H.	Thorpe, A. H.	Woodrow, G.
†Perryman, F. S.	Waller, F. W.	

Part IV.—Section B.

Nine candidates sent in their names, all of whom presented themselves, and seven passed, namely:

†Davis, C. M.	†Perryman, F. S.	†Shrewsbury, A. H.
Hooker, P. F.	†Reynolds, P. C.	
†Mabon, J. B.	†Rowell, A. H.	

Part IV.—Section C.

Seven candidates sent in their names, all of whom presented themselves, and three passed, namely:

†Freeman, H.	†Johnston, W. N.	Taylor, F. G.

Those marked (†) have now completed the Examination for the Class of Fellow.

By Order of the Council,

ALFRED HENRY,
Chairman of Board of Examiners.

H. J. BAKER,
C. R. V. COUTTS,
Joint Honorary Secretaries.

July 1922.

PROCEEDINGS AT THE ANNUAL GENERAL MEETING.

The Seventy-Fifth Annual General Meeting of the Institute of Actuaries was held in Staple Inn Hall, Holborn, on Monday, 12 June 1922, Sir A. W. Watson, K.C.B. (President), in the Chair.

The President expressed the pleasure with which they welcomed the presence at the meeting of Mr. C. A. Elliott, a Fellow of the Institute, resident in Sydney, Australia, and now occupying a professional postition of great importance in Australia. It was always an agreeable duty in that Hall to give a cordial welcome to brethren from the Dominions Overseas when they visited the Institute. He hoped that an opportunity would arise for Mr. Elliott to address the meeting at a later stage.

Proceeding to move the adoption of the Report and Accounts the President said that it was an inevitable feature in a body such as the Institute that they had to record a number of deaths among the Fellows during the past twelve months. The report referred in detail to the great loss which the profession had sustained in the death of Mr. Gordon Douglas. Mr. Douglas was an actuary holding an eminent position in the profession in Edinburgh, and was almost equally well-known in London. Many Fellows had cherished friendships with Mr. Douglas, born of the loyalty and zeal with which he supported them upon occasion, and they felt that the profession as a whole was the poorer for the loss of one who adorned it. Of the other Fellows whose deaths were reported, one, Mr. Sharman, died just before the last annual meeting, and he had referred to his death upon that occasion; Mr. Willis Browne, formerly of the India Office, and Mr. Denmead, of the Estate Duty Office at Somerset House, were in the Civil Service. Mr. Peter, one of the younger Fellows, had undertaken a heavy task in the service of one of the great friendly societies which was in much need of sound actuarial advice. He feared that the task was too great for one whose frame was ill-calculated to bear so heavy a burden, and Mr. Peter had died at a comparatively early age. They had also to regret the loss of Mr. R. A. C. Thomas, who became a Fellow in the year 1899.

Turning to the statistics of progress given in the annual report, it would be seen that so far as the Fellowship and the Associateship were concerned, the numbers had been well maintained, but there was, on the other hand, a notable drop in the number of students. He believed the number was very little more than 50 per-cent of what it was twelve or thirteen years ago. Probably the large reduction in the number of students during the past twelve months was attributable to the effect of the war in postponing the studies of men who, in the ordinary course, would have sat for their examinations between 1914 and 1919. No doubt, in a number of cases, it seemed after that lapse of years to be not worth the trouble, and in certain cases, the decision to abandon the long and wearisome course of preparing for the Institute Examinations had been wisely taken. He was not himself disposed to think that the large reduction in the number of students during the last few years was altogether a matter for regret. The task of qualifying as a Fellow of the Institute of Actuaries was not an easy one; it meant many years of arduous study and much self denial at a time of life when physical activity and pleasures seemed to have their maximum value. When a man was content to carry such burdens as were imposed upon him in his path towards the Fellowship of the Institute of Actuaries there should be something in the way of a prospect of substantial reward for him to look forward to within a reasonable period. But the limitations upon professional practice were real and severe. Actuaries never could hope to be numerically a very strong profession. They might hope to be, and he thought they were, influential as a profession, but the number of prizes available to those who took the course of preparation could never be great, and it was not, on the whole, a bad thing that that those who felt that in their case the burden of continuing their studies would be very great, and the chances of reward

small, should drop out and leave the field clear to the smaller number who might reasonably hope to obtain some professional reward for their labours. Personally, he was indisposed to express any deep regret at the decline in the number of students.

Turning to the accounts, it would be seen that during the year outgo had exceeded income by approximately £300. That was not a very satisfactory state of affairs. He thought it might be attributed in no small measure to the extremely heavy cost of the publications, particularly of the *Journal*. The situation was one which required, and would receive, the careful consideration of the incoming Council, particularly in view of the fact it had been decided to issue three numbers of the *Journal* this year instead of the two numbers to which the Council had rigorously, with very real regret, limited the *Journal* during the past few years. On the other side of the account, for what it might be worth, there was, instead of a substantial depreciation of assets, which for several years in succession it was the unfortunate lot of the Council to have to indicate, a very perceptible appreciation in the assets. It would be observed that on the present occasion the accounts were presented in a new form. After taking eminent advice on the question of the best form of accounts, the Council had separated the prize funds from the general account and had allotted specific investments to the prize funds and to the George Hardy Memorial Fund. Therefore, for the first time, the Fellows would be able to see from the balance sheet precisely what were the investments of the general fund and of the two trust funds. With regard to the trust funds, he thought it might be opportune to draw attention to the important development which the Council had agreed upon during the last session. They had, in the first place, amalgamated the Brown Prize Fund and the Messenger Legacy Fund, and definitely called that combined fund the Prize Fund. The Council had decided that the income of that fund should be distributed in prizes for notable contributions to actuarial science making their appearance either in papers submitted to the Institute or in contributions to the *Journal*, and in accordance therewith a prize of twenty-five guineas had been awarded to Mr. G. S. W Epps for the excellent paper on "Superannuation Funds" that he contributed during the Session 1920–1921. In regard to the Session just closed, the Special Committee charged with the question of examining contributions to actuarial science during the past twelve months had carefully reviewed the papers submitted to the Institute and the articles in the *Journal*, and as a result the Committee had awarded a prize of twenty-five guineas to be divided between Mr. E. H. Lever for his paper on "An Investigation into the Mortality experienced by Life Tenants under Reversions with some conclusions drawn therefrom", and Mr. C. H. Maltby for his paper on "Results of an Investigation into the effect of different Valuation Bases upon Surplus." He sincerely congratulated the prize winners, and asked them to believe that the amount of the prize divided between them was by no means the measure of the appreciation felt for their contributions to actuarial science. It was sincerely to be hoped that they and others would be stimulated to make further efforts, but not merely by the offer of prizes, which after all the Council did not really look upon as an important factor in the case. It was to be hoped also that all the younger members would be stimulated by the obvious appreciation that the Institute expressed at its monthly meetings of the papers which had been submitted, to produce further papers for discussion during the coming and future sessional meetings.

The Council had decided that the income of the George Hardy Fund should be used for the purpose of making grants in connection with research work. The Council was prepared to make grants for work in prospect, for work in progress, and for completed work. It was fully recognized that members of the Institute who might be desirous of undertaking work which involved perhaps a heavy expenditure might be to some extent

deterred from going forward with a subject that might be of great benefit to the profession, and the Council had come to the conclusion that the best use they could make of the George Hardy Memorial Fund—a use which would be the best way of honouring and preserving the memory of the great man after whom the fund was named—was to use the income in grants of the character described. Members contemplating work which they thought might involve an expenditure towards which they would be prepared to accept assistance, had only to notify the honorary secretaries privately, when the matter would be looked into, also privately, by the officers of the Institute and in every suitable case, so far as the means allowed, a grant would be made.

With regard to Examinations held in April last, owing to the large number of candidates who sat, the results were not available when it was necessary to prepare the annual report. They had, however, now become available, and would be announced to the candidates and posted in the Hall. It was a matter of very sincere regret to the Council to find that the number of passes was relatively small when compared with the number of candidates. He was afraid, from the reports of the examiners, that there was still a number of candidates sitting who would be better advised to apply their energies to some other field of study. He was anxious not to discourage men who had a fair prospect of success, but he thought that candidates who had sat for the first examination, Section A of Part I, three or four times should seriously ask themselves whether they were justified in putting themselves to the strain of working for that examination, and of imposing upon the examiners the burden of ascertaining to what extent they had acquired the necessary knowledge. When he looked at the very large number of candidates in Part I, Section A, which was a general mathematical examination, he felt that to some slight extent the examination was tending to become a general scholastic examination in mathematics. He sincerely hoped that those candidates who came forward for the examinations in future, except in very exceptional circumstances indeed, would be men who intended to practise as actuaries either in the outside world or as officers of the great Life Assurance Companies.

He ventured to draw attention to the important fact, that during the coming autumn the Council hoped to issue two new volumes of the *Text-Book* in place of the present Part II. Both Mr. Henry and Mr. Spurgeon had practically finished their labours. Some of the copy was in the hands of the printers, and the Council looked forward to an event of high importance in the issuing of the two new volumes.

The members would recall that the Institute had accomplished one great task on which the hearts of all were set, in the erection of the memorial on the wall of the Institute of their brethren who lost their lives in the war. If no other event had to be recorded that would signalize the year 1921–22. He had no doubt the memorial would hang on the wall of the Institute so long as the Institute existed, and would be a constant reminder to the actuaries of the day of the part that was played by the young members of the actuarial profession in the great war of 1914–1918.

Mr. A. Levine, in seconding the motion, referred to one paragraph in the Report which the President had not mentioned: "The Revised "Regulations and Syllabus of the Examinations to come into force in 1923 "as regards Part I and II, and in 1924 as regards Parts III and IV, "have been issued." He thought that members would agree that the education of the actuary of the future and the examination to be imposed upon him were most important functions of the Institute, and he thought it was quite proper to call attention to the fact that a sub-committee of the Examination Committee was engaged for many months in a most careful consideration of the question, and they hoped the results of that work would prove of great advantage to the Institute.

The motion was carried unanimously.

Election of Officers.

The President announced that in accordance with the by-laws, Mr. Henry Brown, Sir Joseph Burn, Mr. Howard T. Cross, Mr. C. Cosmo Monkhouse snd Mr. Ralph Todhunter had been elected members of the Council, and that the Council and Officers for the ensuing year would accordingly be as follows:

President.

William Peyton Phelps, M.A.

Vice-Presidents.

Abraham Levine, M.A.
William Palin Elderton, C.B.E.
Harold Moltke Trouncer, M.A.
Alfred Charles Thorne.

Council.

Herbert Henry Austin.
Henry James Baker.
Arthur Rhys Barrand, M.P.
Arthur Digby Besant, B.A.
*Henry Brown, B.A.
*Sir Joseph Burn, K.B.E.
Louis Ernest Clinton.
*Howard Turner Cross.
Charles Ronald Vawdrey Coutts.
William Palin Elderton, C.B.E.
Alfred Henry.
Lewis Frederick Hovil.
James Murray Laing.
Abraham Levine, M.A.
George James Lidstone, F.R.S.E.
Harold Edward William Lutt.
Geoffrey Marks, C.B.E.
Reginald George Maudling.
Sir George Ernest May, K.B.E.
Charles Cosmo Monkhouse, B.A., M.C.
Henry John Percy Oakley, M.C.
William Penman.
William Peyton Phelps, M.A.
John Spencer.
Frank Percy Symmons.
Alfred Charles Thorne.
*Ralph Todhunter, M.A.
Harold Moltke Trouncer, M.A.
Samuel George Warner.
Sir Alfred William Watson, K.C.B.

* Not Members of the last Council.

Treasurer.

Arthur Digby Besant, B.A.

Honorary Secretaries.

Henry James Baker.
Charles Ronald Vawdrey Coutts.

Messrs. A. S. Sellar, E. A. Woodall and F. S. Grant were elected auditors for the ensuing year.

Mr. D. C. Fraser proposed a vote of thanks to the President, the Vice-Presidents, the Council and Officers, including the Assistant Secretary, a friend of all the members, for their services during the past year. It was no small task and no small responsibility to maintain the influence of the Institute of Actuaries and the high reputation it enjoyed all over the world and the members gladly recognized that the highest abilities of the profession were freely and zealously given to that duty.

Mr. C. A. Elliott, in seconding the motion, said that before he left Australia he was asked by the Actuarial Society of that country to convey to their brethren in London their fraternal greetings and he was glad to have the opportunity of doing so. Many of the members of the Australian Society were also members of the Institute and they had always

looked on their little Society as in some sense an off-shoot of the Institute. He himself had successively filled the offices of Secretary, Treasurer and President of the Australian Society and had served several years on its Committee and therefore knew exactly what work was involved in the management of a small society, and thus could form a fair estimate of what it must be in the case of the Institute. As one of the private members of the Institute he appreciated the smoothness and efficiency with which the affairs had been managed, and although they in Australia expected no less, owing to the eminence of the gentlemen entrusted with the work of administration, they felt that their debt to them was none the less heavy.

The motion was carried with acclamation.

The President, in reply, thanked the members very sincerely for the kind vote of thanks they had accorded the officials for such services as they had been able to render during the past twelve months. Personally, in retiring from the Chair and the high office to which he had been elected two years ago, he was grateful for the confidence then reposed in him, and was even more grateful for the loyal support he had received during his term of office. It had been a pleasurable and proud experience, and very sincerely had he to thank the members for the honour they had done him and for the manner in which they had sustained him since his election.

A hearty vote of thanks was accorded to the auditors, Messrs. A. S. Sellar, E. A. Woodall and D. M. Carment for their services during the past year.

The President said the time had now arrived for him to hand over the seals of office to his successor, and to ask the members to acclaim Mr. William Peyton Phelps, their new President. The choice of the Council had fallen unanimously on Mr. Phelps for the high office, and that choice he was certain would be endorsed by members, and the whole of the Institute would feel that in Mr. Phelps they had a President who would worthily uphold the dignity and the honour of the Institute and the interests of the profession at large.

Sir Alfred Watson then vacated the Chair, which was taken, amidst cheering, by Mr. Phelps.

The President-Elect, on behalf of the members of the Council that had been just elected, thanked the members for the confidence they had shown in entrusting to them the direction of the affairs of the Institute during the ensuing year. In placing in the hands of the newly-elected Council the control of their interests he thought the members need have no fear but that those interests would be in safe keeping. Many of them had served before in various official capacities and he thought he might say that all had shown in one way or another their devotion to the Institute and their keen interest in whatever tended to advance professional life and work. He was deeply sensible that in conferring on himself the position of President the members had bestowed on him the highest honour they had to give. It was an honour that he valued very greatly indeed, though he found it somewhat difficult to convey his feelings of gratitude. With the advice and assistance of the Council he would do all he could to justify the confidence shown in him, and would endeavour to do the utmost in his power to maintain the high traditions of the responsible office, his election to which he owed to the members' goodwill and not to any merit of his own.

Additions to the Library.

The following works have been added to the Library since the publication of the *Journal* for October 1921 :

By whom presented (when not purchased).

Accountants, Institute of Chartered, in England and Wales.
List of Members, &c., 1922. — *The Institute.*

Accountants and Auditors, Society of.
List of Members, &c., 1922. — *The Society.*

Actuarial Society of America.
Transactions, 1921. — *The Society.*
Containing, *inter alia*—
- "On some of the problems of the smaller Life Insurance Companies", by W. R. Halliday.
- "Industrial Life Insurance", by J. D. Buchanan.
- "Life Insurance without medical examination", by D. E. Kilgour.
- "Insurance without medical examination. Savings in expense compared with expected extra mortality", by A. Hunter.
- "Central Difference Interpolation", by R. Henderson.
- "Causes of total and permanent disability", by J. S. Thompson.
- "Swedish Mortality Investigation, 1921. (Extra Risks)", by J. B. Maclean.
- "Group Mortality Investigation", by E. E. Cammack.
- "Ratings for the principal impairments", by A. Hunter and Dr. O. H. Rogers.
- "Some new problems affecting Life Insurance", by E. B. Morris.
- "Treatment of claims for total and permanent disability", by A. Hunter.
- "An American sickness experience table", by Dr. M. M. Dawson.

Index to Transactions, Vols. I to XV inclusive. 8vo. New York. 1918. — *The Society.*

Actuarial Society of Australasia.
Proceedings, 1921. — *The Society.*

Actuarial Society of Scandinavia.
Transactions, 1921. — *The Society.*

Actuaries, Faculty of.
Transactions, 1921-22. — *The Faculty.*
Containing, *inter alia*—
- "Investments of Insurance Offices. Some practical notes", by R. Murrie.
- "Inaugural Address, 1921", by Lewis P. Orr.
- "Income Tax as affecting Life Assurance Offices", by John Barnett.

	By whom presented (when not purchased).
"Actuary."	
Group Life Assurance. A survey of working problems from the British standpoint. 1922.	*The Publishers.*
American Mathematical Society.	
Transactions.	*The Society.*
American Statistical Association.	
Transactions.	*The Association.*
Association Royale des Actuaires Belges.	
Bulletin of the.	*The Association.*
Association of Life Insurance Medical Directors of America.	
Abstract of Proceedings, 1921–22. 8vo. New York. 1922.	*The Association.*
Bailey (V. A.).	
The mental multiplication and division of large numbers. 1921.	*The Author.*
Belgium.	
Compte Rendu des Opérations et de la situation de la Caisse Générale d'Epargne et de Retraite. 1915–18, 1919, 1920. Statistique des Accidents du Travail. Exposé des Résultats. 1910. La. 4to. Brussels. N.D.	*The Belgian Government.*
Beneś (J.).	
Pensijne pojisteni (Pension Insurance). Prague. N.D. **Beneś (J.) and Prof. Kolousek.** Duchod (On Annuities). Prague. N.D.	*J. Beneś.*
"Biometrika".	*Purchased.*
Vol. XIII, Part IV and Vol. XIV, Parts I and II. Containing, *inter alia*— "On Polychoric coefficients of correlation", by Prof. Karl Pearson. "On expansions of Tetrachoric Functions", by J. Henderson. "Is Tuberculosis to be regarded from the Aetiological standpoint as an acute disease of childhood?", by Dr. A. F. Andvord.	
Brand (The Hon. R. H.).	
War and National Finance. 8vo. 1921.	*Purchased.*
Brownlee (Dr. J.).	
The use of Death Rates as a measure of hygienic conditions. 8vo. 1922.	*The Author.*
Cannan (E.).	
Wealth. A brief explanation of the causes of economic welfare. 2nd. edit. 8vo. 1920.	*Purchased.*

By whom presented (when not purchased).

Casualty Actuarial Society of America.

Proceedings, 1921. — *The Society.*

Containing, *inter alia*—

"Industrial retirement systems based on the money-purchase principle", by J. H. Woodward.

"Classification of risks as the basis of insurance rate making, with special reference to Workmen's Compensation insurance", by A. H. Mowbray.

"Remarriage experience of Pennsylvania Compensation Insurance Carriers, policy years 1916–1919", by E. H. Downey.

"Mortality from external causes among Industrial Policy-holders of the Metropolitan Life Insurance Co., 1911–1920", by Dr. L. I. Dublin and E. W. Kopf.

"Observations on Pension Funds for Employees rendered permanently disabled by a second injury", by A. H. Mowbray.

Cathles (L. M.).

The actuary's function in drafting the Life Insurance contract. Dallas, Texas. 1921. — *The Author.*

Czecho-Slovakia.

Fünfundzwanzig Jahre Arbeiter-Unfall-Versicherung. 1889–1914. 4to. Prague. 1915. — *J. Benés.*

Chartered Insurance Institute, Journal of the.

Vol. XXIV. 8vo. 1921. — *The Institute.*

Containing, *inter alia*—

"The effect of War on Life Assurance", by H. Brown.

"Elementary notes on Titles to Life Policies", by Dr. A. E. Sprague.

"Some everyday problems in Life Assurance from the medical standpoint", by Dr. R. Scott Skirwing.

Cohen (J. L.).

Insurance against Unemployment, with special reference to British and American conditions. 8vo. 1921. — *Purchased.*

Czuber (Dr. E.).

Die Statistischen Forschungsmethoden. 8vo. Wien. 1921. — *Purchased.*

Denmark.

Beretning fra Forsikringsraadet for aaret 1921. — *The Danish Government.*

Deutscher Verein für Versicherungs-Wissenschaft.

Zeitschrift, 1921–1922. — *The Society.*

Economic Society (Royal).

Journal of the, 1921–22. — *Purchased.*

By whom presented (when not purchased).

Edwards (J.).	
Differential Calculus for beginners. 8vo. 1908. Integral Calculus for beginners, with an introduction to the study of Differential Equations. 8vo. 1907.	*E. H. Kemp.*
Fisher (Arne).	
The Mathematical Theory of Probabilities and its application to Frequency Curves and Statistical Methods. Vol. I. 2nd. edit. 8vo. New York. 1922.	*Purchased.*
France.	
Annuaire des Sociétés d'Assurances Opérant en France, et des Compagnies étrangères, 1922.	*The Editor.*
Fraternal Actuarial Association.	
Proceedings, 1920-21.	*The Association.*
Gough (G. W.).	
Wealth and Work. Sm. 8vo. 1921.	*Purchased.*
Grieshaber (Dr. H.).	
Beitrage zur Kontinuierlichen Methode in der Krankenversicherung. 8vo. Bern. 1919.	*The Author.*
Guardian Assurance Co.	
A Record of the, 1821–1921. By A. W. Tarn and C. E. Byles. 8vo. 1921.	*A. W. Tarn.*
Guldberg (A.).	
On Correlation. Cristiania. 1921. On the Correlation of successive operations. Cristiania. 1921,	*The Author.*
"Hafnia" Life Insurance Co.	
Jubilee Volume, 1872–1922. 4to. Copenhagen 1922.	*The Company.*
Hawtrey (R. G.).	
The Exchequer and the Control of Expenditure. 8vo. 1921.	*Purchased.*
Holland.	
Vereeniging voor de Verzekerings-Wetenschap. Het Verzekerings-archief.	*The Society.*
Hunter (A.).	
Treatment of Claims for permanent total disability. New York. 1921. Insurance without medical examination. New York. 1921.	*The Author.*
Hunter (A.) and Dr. O. H. Rogers.	
Ratings for the principal impairments. New York. 1921.	*A. Hunter.*
Insolera (F.).	
Sulla Geometria delle Operazioni di Borsa. Torino. 1921.	*The Author.*

By whom presented (when not purchased).

Institut des Actuaires Français.

Bulletin of the. — *The Institute.*

Institute of Actuaries Students' Society.

Journal, Vol. II, No. I. — *Purchased.*

Containing, *inter alia*—

"Surrender-values and paid-up policies", by V. W. Tyler.

"Reversions and Life Interests", by E. H. Lever.

"Problems of the 1921 Census", by R. Levey.

"Discounted Bonus Policies", by W. E. Hustwitt.

"Exchange fluctuations and the present financial position, with special reference to Indian finance", by A. J. C. Edwards.

"National Insurance (Health) Acts, 1911-18", by O. C. J. Klagge.

Institute of Bankers.

Journal of the, 1921–22. — *The Institute.*

Insurance Institute of Toronto.

Proceedings, 1921–22. — *The Institute.*

Containing, *inter alia*—

"Health and Accident features in Life Policies" by H. R. Stephenson.

"Life Insurance without medical examination", by E. E. Reid.

"Nature of Life Insurance. Its basic principles and uses", by C. P. Muckle.

"The History and Development of personal accident and sickness insurance", by F. J. Lightbourn.

International Actuarial Congresses.

Bulletin of the Permanent Committee. 1921. — *The Permanent Committee.*

International Labour Office (League of Nations).

International Labour Review. Oct. 1922. — *The Editor.*

Italy.

Giornale di Matematica Finanziaria Rivista technica del Credito e della Previdenza. 1921. — *F. Insolera.*

Istituto Nazionale delle Assicurazioni. Situazione al 31 Dec. 1917 and 1920. Fol. Rome. 1922.

Regarding the Report of the "Assicurazioni Generali di Venezia" and the "Compagnia Adriatica di Sicurta". Rome 1921–22. — *Comm. Ing. Guido Toja.*

Jessop (C. M.).

Elementary Analysis. 8vo. Camb. 1921. — *Purchased.*

Kapteyn ((J. C.).

Skew frequency curves in Biology and Statistics. Groningen. 1916. — *Purchased.*

Keller (Dr. Max).

Die Behandlung des Kriegsrisikos in der Lebensversicherung unter dem Einfluss des Weltkrieges. Berlin. 1922. — *The Author.*

By whom presented (when not purchased).

Keynes (J. M.).
A Revision of the Treaty, being a sequel to "The Economic Consequences of the Peace." 8vo. 1922. — *Purchased.*

Kirkaldy (A. W.)—Editor.
British Finance during and after the War. 1914-21. 8vo. 1921. — *Purchased.*

Le Mesurier (L.).
Common-sense Economics. 8vo. 1922. — *Purchased.*

Lister (Dr. T. D.).
Medical Examinations for Life Insurance. 8vo. 1921. — *Purchased.*

London Mathematical Society.
Proceedings, 1921–22. — *The Society.*

Ministry of Health.
Reports of the Medical Research Committee, 1921–22.
Report of the Chief Medical Officer, 1921. — *Purchased.*

Norway.
Forsikringsselskaper, 1919, 1920. 8vo. Christiania. 1921–22. — *The Norwegian Government.*

Palmer (A. R.).
The use of Graphs in commerce and industry. Sm. 8vo. 1921. — *Purchased.*

Parliamentary Papers.

Acts of Parliament.
Census, 1920. — *Purchased.*

Assurance Companies.
Returns to the Board of Trade. 1921. — *Purchased.*

Censuses.
1921, Scotland. Report, Vol. I, Parts I–VI. Fol. 1922. — *The Registrar-General.*
1921. India. Baroda State. Imperial and State Tables. Fol. Bombay. 1921. — *The Government of Bombay.*

Colonies.

Canada.
Report of the Superintendent of Insurance for the year 1920. — *The Superintendent of Insurance.*
Insurance Companies. Abstract of Statements for the year ended 31 Dec. 1921. 8vo. 1922. — *The Canadian Government.*

New South Wales.
Friendly Societies, &c. Report of the Registrar for 1921.
Official Year Book, 1921.
Report of the Government Statistician on Vital Statistics, 1921.
Report by the Government Statistician on the Vital Statistics of the Metropolitan combined sanitary districts, 1921.
State Superannuation Board. Report, 1921.
Statistical Register for 1920–21 and previous years. — *The Government of N.S.W.*

	By whom presented (when not purchased).
Parliamentary Papers—*continued.*	
Colonies—*continued.*	
New Zealand.	
Friendly Societies Annual Report of the Registrar, 1920.	
Government Insurance Department. Annual Report, 1920.	
National Provident Fund, Annual Report, 1920. Actuarial examination for the period ended 31 Dec. 1919.	*The Government of N.Z.*
Official Year Book, 1921–22.	
Report on the Results of the Census of 1921. Part I. Population.	
Statistics of the Dominion for the year 1920.	
South Australia.	
Friendly Societies. Report of the Public Actuary, 1919.	*The Public Actuary.*
Victoria.	
Friendly Societies. Forty-third Annual Report of the Government Statist, 1920.	*The Government of Victoria.*
Western Australia.	
Friendly Societies. Reports of Proceedings by the Registrar for the years ended 30 June 1921 and 1922.	*The Government of W.A.*
Statistical Register, 1920–21.	
Income Tax regulations for Superannuation Funds. 1921.	*Purchased.*
India.	
Life Assurance Companies. Returns of Companies doing business in British India. Fol. Simla. 1921.	*H. G. W. Meikle.*
National Expenditure. Report of Committee on. 3 Parts. 8vo. 1922.	*Purchased.*
National Health Insurance. Report by the Government Actuary on the valuation of the Assets and Liabilities of Approved Societies as at 31 Dec. 1918. 8vo. 1922.	*Purchased.*
Registrar-General. England. Eighty-third Annual Report of Births, Deaths and Marriages, 1920.	*The Registrar-General.*
Pearson (Prof. Karl).—Editor.	
Tracts for Computers, No. IV. Tables of the Logarithms for the complete Γ-function to 12 figures. Originally compiled by A. H. Legendre. 4to. 1921.	*Prof. Pearson.*
No. VI. Smoothing, by E. C. Rhodes. 8vo. 1921.	*The Author.*
Periodicals.	
American Economic Review.	*The Editor.*
Bankers' Magazine.	*Purchased.*
Economist.	*Purchased.*
Insurance Record.	
Journal of Political Economy. (Chicago).	*The Editor.*
Post Magazine.	
Post Magazine Almanack.	
Revue du Travail. (Brussels).	*The Belgian Government.*
The Secretary.	*Institute of Secretaries.*

By whom presented (when not purchased).

Pokorny (M.).

Duchod Invalidini (On Annuities for invalidity). Prague. 1885. — *J. Beneš.*

Polish Institute of Actuaries.

Journal of the, 1922. — *The Institute.*

Schütze (H.).

Die Mathematischen Grundlagen der Lebensversicherung. Sm. 8vo. 1922. — *The Publisher.*

Scientific and Learned Societies of Great Britain and Ireland.

Year Book, 1921. — *Purchased.*

Soper (H. E.).

Frequency Arrays, illustrating the use of logical symbols in the study of statistical and other distributions. Camb. 1922. — *Purchased.*

Stamp (Sir Josiah), K.B.E.

Wealth and Taxable capacity. 8vo. 1922. — *Purchased.*

Statistical Society (Royal).

Journal of the, 1921–22.
Catalogue of the Library. 8vo. 1921. — *The Society.*

Steffensen (Dr. J. F.).

On the remainder form of certain formulas of mechanical quadrature. Copenhagen. 1921.
On numerical integration of Differential Equations. Copenhagen. 1922.
A correlation formula. Copenhagen. 1922. — *The Author.*

Sweden.

Enskilda Försäkringsanstalter ar 1920, av Kungl. Försäkringsinspektionen. — *The Swedish Government.*

Switzerland.

Rapport du Bureau Fédéral des Assurances sur les Entreprises privées en matière d'Assurances en Suisse. 1919. — *The Swiss Government.*

United States of America.

Official Publications.
Twentieth and Twenty-first Annual Reports of the Bureau of the Census on Mortality Statistics, 1919, 1920.
Fifth and Sixth Annual Reports on Birth Statistics for the Birth Registration area of the United States, 1919, 1920. — *The U.S. Government.*

Connecticut.
Fifty-seventh Annual Report of the Insurance Commissioner, 1921. — *The Commissioner.*

Massachusetts.
Sixty-sixth Annual Report of the Insurance Commissioner (Life and Miscellaneous), 1920. — *The Commissioner.*

New York.
Reports of the Superintendent of Insurance, 1920, 1921. — *The Superintendent.*

By whom presented (when not purchased).

Veley (Dr. V. H.), F.R.S.

Depreciation. 8vo. 1922. *Purchased.*

Wales (William).

An Inquiry into the present state of Population in England and Wales. 8vo. 1781. *Purchased.*

Webster (Alexander).

An account of the number of People in Scotland in the year 1755. La. Fol. N.D. *Dr. J. C. Dunlop.*

Yule (G. Udny).

The fall of the Birth-rate. Camb. 1920. *Purchased.*

The following books were presented to the Library by Mr. Stanley Day, F.I.A. As more than one copy of the majority of these books was already in the Library of the Institute, discretion was accorded the Hon. Librarians to deal with the volumes as they thought fit. Those not required by the Institute were accordingly distributed amongst other bodies with Mr. Day's consent :

Ansell (C.).

A Treatise on Friendly Societies. 8vo. 1835.

Ansell (C.), Jr.

Statistics of Families in the Upper and Professional Classes. 8vo. 1874.

Assurance Pamphlets.

Miscellaneous, including two bound volumes. 8vo. V.D.

Babbage (C.).

Comparative View of the various institutions for the Assurance of Lives. 8vo. 1826.

Table of Logarithms of the natural numbers from 1 to 108000. 4th impression. 8vo. 1844.

Baily (F.).

Tables for the purchasing of Leases for terms of years certain, and for Lives, &c. 2nd edit. 8vo. 1807.

The Doctrine of Life Annunities and Assurances analytically investigated, &c. Vol. I. 8vo. 1813.

The Same, by H. Filipowski. 8vo. 1864.

Barlow (P.).

New Mathematical Tables containing the Factors, Squares, Cubes, Square-roots, Reciprocals and Hyperbolic Logarithms of all numbers from 1 to 10000. 8vo. 1814.

Brown (S.). Peter Hardy and Col. J. T. Smith.

Report on the Madras Military Fund. 8vo. 1863.

Chisholm (J.).

Tables for finding the value of policies of all durations, according to any Table of Mortality or any Rate of Interest, with other useful Tables. 4to. 1885.

Corbaux (F.).

The Doctrine of Compound Interest, with new and compendious Tables. Roy. 8vo. 1825.

Dale (M.).

Calculations deduced from first principles, by plain arithmetic, for the use of the Societies instituted for the benefit of old age. 8vo. 1772.

Davies (G.).

Treatise on Annuities (posthumous). 8vo. N.D.

Tables of Life Contingencies, containing the values of annuities and assurances on the experience of the Equitable Society, and other Tables. 8vo. 1825.

De Moivre (A.).

Annuities on Lives. 3rd edit. 8vo. 1750.

De Morgan (A.).

Arithmetic and Algebra (1827). Examples of the processes of Arithmetic and Algebra (1847). Elementary illustrations of the Differential and Integral Calculus (1842). In one vol. 8vo.

Equitable Life Assurance Society.

The Deed of Settlement, with Bye-laws, and four Addresses. In one vol. 8vo. 1811.

Filipowski (H.).

A Table of Anti-Logarithms, containing to seven places of decimals, natural numbers answering to all numbers from ·00001 to ·99999, and an improved Table of Gauss' Logarithms, &c. 2nd edit. 8vo. 1851.

Galloway (T.).

A Treatise on Probability. 8vo. Edin. 1839.

Gray (P.).

Tables and Formulæ for the computation of Life Contingencies. 8vo. 1849.

Gray (P.), H. A. Smith and W. Orchard.

Assurance and Annuity Tables. Carlisle 3 per-cent. 8vo. 1851.

Jones (D.).

Value of Annuities and Reversionary Payments, with Tables. 2 vols. 8vo. 1843.

Jones (J.).

A series of Tables of Annuities and Assurances calculated from a new rate of mortality amongst Assured Lives. 8vo. 1843.

Laundy (S. L.).

Table of Quarter Squares of all integer numbers up to 100,000. Roy. 8vo. 1856.

M'Kean (A.).

Practical Life Tables Chart. 8vo. 1861.

Milne (J.).

A Treatise on the Valuation of Annuities and Assurances on Lives and Survivorships. 2 vols. 8vo. 1815.

Morgan (W.).

The Doctrine of Annuities and Assurances on Lives and Survivorships, &c. 8vo. 1779.

Orchard, (W.).

Single and annual Premiums for every value of annuity adapted to any rate of mortality. 8vo. 1850.

"Post Magazine".

1848–1852. 2 vols. 4to.

Price (R.).

Observations on Reversionary Payments, on schemes for providing Annuities for Widows, &c. 7th edit. by W. Morgan. 2 vols. 8vo. 1812.

The Same. Vol. I only.

Rock Life Assurance Co.

The Deed of Settlement, Bye-laws and an Address. 8vo. 1834.

Scottish Widows' Fund & Life Assurance Society.

Deed of Constitution, Bye-laws, Reports and Addresses. Edinb. 8vo. 1834.

Sprague (Dr. T. B.).

Treatise on Life Insurance Accounts. 8vo. 1874.

Sutton (W.).

Report on the Bengal Civil Fund, including valuations as at 31 March 1873 and 31 March 1878. 8vo. 1879.

Wood (J.).

The Elements of Algebra. 15th edit. by T. Lund. 8vo. 1857.

THE INSTITUTE OF ACTUARIES.

Memorandum as to Alterations in Examination Syllabus.

The Council of the Institute having adopted a revised Examination Syllabus, to come into force in 1923 *as regards Parts* I *and* II, *and in* 1924 *as regards Parts* III *and* IV, *it has been thought desirable to issue a short statement as to the changes that have been made.*

The general scheme of the revised Syllabus is to examine candidates in three groups of subjects, namely :

(*a*) purely mathematical subjects ;

(*b*) subjects involving the application of mathematical theory to actuarial questions ;

(*c*) practical questions arising out of the subjects previously studied, and other questions not strictly actuarial but of which a knowledge is required by the actuary in the practice of his profession.

Part I.

This Part consists of one Section only. The mathematical portion is unchanged, but questions may be set involving a knowledge of the nature and elementary relations of trigonometrical functions and of the methods of graphs and rectilinear and polar co-ordinates. The subject of the Elements of Statistics has been introduced.

Part II.

The subject of Compound Interest and Annuities-Certain, formerly Section B of Part I, now becomes Section A of this Part. Section B covers the same ground as Part II of the previous Syllabus.

Part III.

Section A of this Part is practically identical with Section A of Part III of the previous Syllabus. Section B includes subjects which were contained in Section B of Part III and Sections B and C of Part IV of the previous Syllabus, namely, the Valuation of the Contracts of Life Assurance and Employers' Liability Insurance Companies; the Valuation of, and Calculation of Rates of Contribution for, Friendly Societies, Pension Funds, and Widows' and Orphans' Funds; and the Calculation of Office Rates of Premium for Assurances and Annuities.

Part IV.

The following subjects which were included in Section B of Part III and Sections A and C of Part IV of the previous Syllabus have been transferred to this Part, namely the Distribution of Surplus; Life Interests and Reversions; the Acts relating to Life Assurance and Employers' Liability Insurance Companies; and Official Statistics. The subject of Investments has been introduced.

Note.—The Examinations in Parts III and IV to be held in 1923 will be conducted under the Syllabus dated March 1918.

By Order of the Council,

A. C. THORNE, }
H. J. BAKER, } *Hon. Secs.*

April 1922.

REGULATIONS and SYLLABUS of Examinations for admission to the Classes of Student, Associate, and Fellow.

(*These Regulations will come into force, and cancel all previous Regulations, in* 1923 *as regards Parts* I *and* II, *and in* 1924 *as regards Parts* III *and* IV.)

1. The Examinations held by the Institute are four in number, distinguished as Parts I, II, III, and IV respectively. They will, until further notice, be conducted in writing, at such places and under such conditions as the Council may prescribe.

2. The subjects of the Examinations are as set out in the annexed Syllabus.

3. The names of successful candidates in each Part, or Section of a Part, will be arranged in alphabetical order, without distinction of Class.

4. No candidate will be allowed to present himself for Examination until he has paid all Entrance Fees, Subscriptions, and Examination Fees that may be due, and complied with the requirements of the Bye-laws and of these Regulations.

5. Examinations will be held in May of each year, or at such other times as the Council may prescribe.

6. No candidate shall present himself for Examination in any Part of the annexed Syllabus until after passing the previous Part, except that :

(*a*) a graduate in Mathematical Honours of any University in the British Empire, or any other candidate on the recommendation of a Tutor of the Institute or of two Fellows having personal knowledge of his qualifications, may present himself for Examination in Section A of Part II at the same time as Part I, provided that, should he fail in Part I, he will be deemed not to have presented himself for Section A of Part II ;

(*b*) a candidate who has passed Part I may present himself for Examination in both Sections of Part II at the same time, provided that, should he fail in Section A, he will be deemed not to have presented himself for Section B ;

(*c*) a candidate may take both Sections of Part III at the same Examination, or either Section may be taken at one Examination, and the remaining Section at a subsequent Examination, but he must pass in both Sections.

7. The Examination Fee payable for each Part of the Syllabus is £1 11*s*. 6*d*., and for one Section of a Part £1 1*s*.

8. In these Regulations the expression " those who have passed " any particular Part or Section of a Part shall be construed as including any persons who under the Bye-laws or any previous Regulations shall have been exempted from passing that particular Part or Section of a Part.

9. Candidates who have passed the prescribed Examinations will be admitted to the Class of Student, Associate, or Fellow after signing the proper Form of Obligation or of Transfer, as the case may be, and paying the Subscription of the Class for the current year.

10. At least four months' notice will be given by public advertisement of the places at which Examinations will be held and of the dates of such Examinations.

11. Candidates for Examination at any place in the United Kingdom must give notice in writing to the Assistant Secretary so that such notice shall reach the Assistant Secretary at least fourteen days before the date of Examination, and a candidate for Examination at any place outside the United Kingdom must give such notice so that it shall reach the Assistant Secretary at least two months before the date of Examination. At the time of giving notice a candidate must specify the Part or Section of a Part for which he intends to present himself, and must pay the prescribed fee.

Class of Student.

12. An applicant for admission to the Class of Student must (*a*) have furnished such evidence of general education as the Council may from time to time prescribe (see § 13) and (*b*) have passed Part I of the Examinations under these or previous Regulations.

13. As evidence of general education the Council will require a certificate showing that the applicant has passed the Matriculation or General School Examination of the University of London, or a similar Examination of any University in the British Empire, or the Oxford or Cambridge Senior Local Examination, or some other Examination approved by the Council from time to time or accepted on individual application.

Class of Associate.

14. Candidates for admission to the Class of Associate shall be members of the Class of Student, and shall be required to pass Parts I, II, and III of the annexed Syllabus, provided that :

(*a*) those who have passed Part I of any previous Syllabus, or Section A only of Part I of the Syllabus dated March 1918, will be deemed to have passed Part I of the annexed Syllabus ;

(*b*) those who have passed Section B only of Part I of the Syllabus dated March 1918 will be deemed to have passed Section A of Part II of the annexed Syllabus, but they will be required to pass Part I of the annexed Syllabus before presenting themselves for Section B of Part II ;

(*c*) those who have passed both Sections of Part I of the Syllabus dated March 1918 will be required to pass Section B only of Part II of the annexed Syllabus ;

(*d*) those who have passed Parts I and II of any previous Syllabus will be deemed to have passed Parts I and II of the annexed Syllabus ;

(*e*) those who have passed Parts I, II, and Section A of Part III of the Syllabus dated March 1918 will be deemed to have passed Parts I, II, and Section A of Part III of the annexed Syllabus ;

(*f*) those who have passed Parts I, II and III of the Syllabus dated March 1918 will be deemed to have passed Parts I, II and III of the annexed Syllabus.

(*g*) those who have passed Part I of any previous Syllabus and will pass Part II not later than May 1923 will be admitted to the Class of Associate under the previous Regulations ;

(*h*) Fellows of the Faculty of Actuaries by Examination who have been, or who may hereafter be, elected Associates of the Institute, will be deemed to have passed Parts I, II, and III of the previous Syllabus or of the annexed Syllabus.

Class of Fellow.

15. Candidates for admission to the Class of Fellow shall be members of the Class of Student or Associate as defined above, and, except as hereinbefore provided, shall be required to pass Parts I, II, III, and IV of the annexed Syllabus.

March 1922.

Regulations for Probationers.

As a necessary preliminary to admission to the Class of Student, application to become a Probationer must be made on a form prescribed for the purpose, and the applicant must furnish to the Council such evidence of his general education as is required under § 13. If the application be approved the applicant shall become a Probationer on payment of an Entrance Fee of 10*s*. 6*d*., but the Council may at any time withdraw their approval, and thereupon he shall cease to be a Probationer. Should a Probationer subsequently be admitted a Member of the Institute, the fee of 10*s*. 6*d*. paid by him on becoming a Probationer, will be taken as paid on account of the Entrance Fee as Student.

The annual Subscription for Probationers is 10*s*. 6*d*., payable on admission and on 1 October in each year. If the subscription for any year be not paid before the 31 December, then the defaulter shall no longer be a Probationer.

Probationers, while not being Members of the Institute, are allowed the following privileges, namely :

They are entitled to join the classes for Students, in accordance with the rules prescribed for such classes, and to attend the Ordinary General Meetings of the Institute, but not to vote or take part in the discussions thereat.

They may borrow books from the library for the purposes of their studies, but this privilege is subject to the discretion of the Librarians, and to the rules which the Council may from time to time prescribe.

SYLLABUS OF EXAMINATIONS.

Part I.

(1) Advanced Algebra, including Logarithms and Probabilities.

(2) Finite Differences, including Interpolation and Summation; Elementary Differential and Integral Calculus.

(3) Elements of Statistics.

Questions may be set involving a knowledge of the nature and elementary relations of trigonometrical functions and of the methods of graphs and rectilinear and polar co-ordinates.

Part II.

Section A.

Compound Interest and Annuities-Certain, including the construction and use of relative Tables.

Section B.

The Theory of Life Contingencies and other Contingencies within the scope of actuarial work. The construction of the Life Table and other Tables based on such contingencies (excluding compilation of Tables from Statistics and Graduation) and of functions based thereon; including the values of Annuities, Assurances (of various types) and Net Premiums.

Part III.

Section A.

(1) Compilation of Tables from Mortality, Sickness, Accident and other similar Statistics. Graduation of such Tables.

(2) History and Distinctive Features of the principal Mortality and Sickness Tables now in general use.

Section B.

(1) Calculation of Office Rates of Premium for Assurances and Annuities depending on Life and other Contingencies, and of Rates of Contribution for Sickness, Pension and Widows' and Orphans' Funds.

(2) Valuation of the Contracts of Life Assurance and Employers' Liability Insurance Companies, Friendly Societies, Pension Funds and Widows' and Orphans' Funds.

Part IV.

(1) Analysis and Distribution of Surplus and other practical problems arising on the valuation of Life Assurance and Employers' Liability Insurance Companies, Friendly and Approved Societies, Pension Funds and Widows' and Orphans' Funds.

(2) Valuation of Life Interests and Reversions. Surrender Values of Policies and other Contracts.

(3) The application of Statistical Methods and the employment of Official Statistics in the solution of actuarial problems.

(4) The Investment of the Reserve Funds of Insurance and other Societies; the valuation of investments.

(5) General provisions of the Acts relating to Life Assurance Companies, Employers' Liability Insurance Companies and Friendly and Approved Societies, including the relative schedules and method of compiling them.

COURSE OF READING

recommended for the guidance of Students in connection with the Examinations.

The latest editions of Text-Books are referred to in each case. The references to Transactions of Actuarial Societies have been restricted with the object of directing special attention to particular phases of the subjects. No responsibility is accepted for any opinions expressed in the papers recommended or in the discussions following the papers, which should also be studied.

In addition to the subjects in the suggested course of reading Candidates should also be familiar with current actuarial and insurance topics.

Part I.

(1) H. S. Hall and S. R. Knight: "Higher Algebra." (Macmillan and Co., Ltd.). Students need not read the parts of the book relating to Interest and Annuities, advanced Convergency and Divergency of Series, Continued Fractions, Indeterminate Equations of the second degree, Theory of Numbers, Probabilities, Determinants, Elimination or Cubic and Biquadratic Equations.

(2) A. Henry: "Calculus and Probability for Actuarial Students." (C. & E. Layton.)

(3) W. P. and E. M. Elderton: "Primer of Statistics." (A. and C. Black.)

Part II.

(1) Institute *Text-Book*, Part I (except Chapters ix and x).

(2) E. F. Spurgeon: "Life Contingencies." (C. & E. Layton.)

Part III.

Section A (1) and (2).

W. P. Elderton and R. C. Fippard: "The Construction of Mortality and Sickness Tables." (A. and C. Black.)

J.I.A.,* vol. xlvii, pp. 66-86 (Mortality of Government Life Annuitants).

Supplement to the 75th Annual Report of the Registrar-General, Part I, Life Tables (Cd. 7512, 1914). (The Review of this Report, *J.I.A.*, vol. xlix, p. 96, should be read as an introduction to the Report).

J.I.A., vol. xlix, p. 297, (G. King: The New National Life Tables).

Sir A. W. Watson: "Account of an Investigation of the Sickness and Mortality Experience of the I.O.O.F., Manchester Unity, 1893-97", pp. 1-80. (C. & E. Layton.) *See also J.I.A.*, vol, xxxviii, pp. 369-72, 533-4, commenting thereon.

J.S.S.† Special Number on the Construction and Graduation of Tables.

Sir George F. Hardy: "The Theory of the Construction of Tables of Mortality, &c.", chapters i and ii, and pp. 63-70. (C. & E. Layton).

J.I.A., vol. xxx, p. 212 (G. J. Lidstone: Graduation of a limited Experience by reference to a Standard Table); vol. xxxviii, p. 11 (G. J. Lidstone: Makeham's Law); vol. xxxviii, p. 501 (Sir G. F. Hardy: Graduation of the $O^{[NM]}$ Table); vol. xli, p. 348 (G. J. Lidstone: Graduation by Summation); vol. xlvii, p. 548 (Extract from Cd. 6907, 1913, National Insurance Mortality Tables).

J.S.S., vol. i, No. 2, p.44 (W. P. Elderton: Normal Curve of Error and Average Deviation).

Part III.

Section B (1) and (2).

T.F.A.,‡ vol. ii. p. 207 (H. Moir: Office Premiums).

Insurance Guide and Handbook, vol. i, chapters v and vi. (C. & E. Layton.)

J.S.S., Special Number on Extra Risks.

J.S.S., vol. i, No. 5, p. 7 (R. C. Simmonds: Pension Fund Valuations).

* *Journal of the Institute of Actuaries.*

† *Journal of the Institute of Actuaries Students' Society.*

‡ *Transactions of the Faculty of Actuaries.*

J.I.A., vol. xxxix, p. 129 (G. King: Staff Pension Funds); vol. xxxix, p. 337 (S. J. H. W. Allin: Social Conditions as affecting Widows' and Orphans' Funds); vol. xl, p. 188 (E. C. Thomas: Some Special Features of Widows' and Orphans' Funds): vol. xl, p. 200 (O. Schjoll: The Calculation of the Contributions, &c.).

J.I.A., vol. xxvii, pp. 270-81 (Sir G. F. Hardy: Friendly Societies).

Sir A. W. Watson: "Account of an Investigation of the Sickness and Mortality Experience of the I.O.O.F., Manchester Unity, 1893-97, pp. 80-135. (C. & E. Layton.)

J.I.A., vol. xxxvii, p. 57 (S. G. Warner: Net Premium Method of Valuation); vol. xxxviii, p. 1 (G. J. Lidstone: Valuation of Endowment Assurances); vol. xxxviii, p. 385 (D. C. Fraser: Comparison of Valuation Methods); vol. xl, p. 122 (D. C. Fraser: Use of $O^{[M]}$ Premiums for Valuation purposes); vol. xlii, p. 145 (G. King: Altenburger's Method of Valuation); vol. xlii, p. 409 (G. J. Lidstone: Notes on Valuation Methods); vol. xlviii, p. 1 (W. P. Elderton: Approximate Valuation of Endowment Assurances); vol. xlviii, p. 121 (A. E. King: An Extension of existing Valuation Methods); vol. li, p. 118 (A. Henry: On a Method of Approximate Valuation).

T.F.A., vol. vi, p. 93 (A. Fraser: Valuation of Limited-Payment Policies).

J.S.S. vol. i, No. 2, p. 24 (E. A. Woodall: Valuation of Special Policies); vol. i, No. 2, p. 31 (R. C. Fippard: Valuation of Industrial Assurance Policies).

J.I.A., vol. l, p. 231 (P. H. McCormack: Continuous Valuation Machinery).

Journal of the Chartered Insurance Institute, vol. xi, p. 303 (R. M. M. Roddick: Valuations under Employers' Liability Insurance Companies Act, 1907).

J.I.A., vol. xlv, p. 101 (W. Penman: Valuation of Liabilities under Employers' Liability Contracts).

Part IV.

Candidates should possess a general knowledge of current financial conditions, and they should also be familiar with the financial provisions of the National Health Insurance Acts.

J.I.A., vol. xxxviii, pp. 80-88 (Sir G. H. Ryan: Methods of Valuation and Distribution of Profits); vol. xxxvii, p. 453 (G. King: Comparative Reserves); vol. xli, p. 18 (J. Buchanan: Comparative Reserves); vol. xlii, p. 161 (C. R. V. Coutts: Bonus Reserve Valuations).

J.S.S., vol. i, No. 4, p. 5 (R. C. Simmonds: Analysis of Profits); Special Number on Valuation of Liabilities and Distribution of Surplus.

J.S.S., vol. i, No. 5, p. 18 (Editorial).

J.I.A., vol. lii, p. 405 (G. S. W. Epps: Superannuation Funds).

Sir A. W. Watson: "Lectures on Friendly Societies." (C. & E. Layton).

J.I.A., vol. xxvii, pp. 245-63, 270-81, 293-348 (Sir G. F. Hardy: Friendly Societies).

Actuaries' Report on the Schemes embodied in the National Insurance Bill, 1911 (*J.I.A.*, vol. xlv, p. 406).

Reports on the Administration of National Health Insurance (1912-13, Cd. 6907, pp. 16-35 and 552-601; 1913-14, Cd. 7496, pp. 29-64,; 1914-17, Cd. 8890, pp. 7-15); Review, *J.I.A.*, vol. xlviii, p. 109.

Reports of the Departmental Committee on Approved Society Finance and Administration (Cd. 8251, Cd. 8396, Cd. 8451); Reviews, *J.I.A.*, vol. l, pp. 99, 221.

J.S.S., vol. i, No. 4, p. 50 (Editorial); vol. i, No. 5, p. 24 (Editorial); Special Number on Reversions.

J.I.A., vol. liii, p. 1 (E. H. Lever: Mortality Experience of Life Tenants under Reversions); vol. xxxvi, p. 81 (Correspondence with Inland Revenue).

T.F.A., vol. iii, p. 201 (A. E. Sprague: Paid-up Policies and Surrender Values).

A. L. Bowley: "Official Statistics" (Humphrey Milford). *Additional reading on this subject will be announced later.*

Supplement to the 65th Annual Report of the Registrar-General (Cd. 2618, 1907, Cd. 2619, 1908); Review, *J.I.A.*, vol. xliii, p. 230.

T.F.A., vol. v, p. 1 (J. C. Dunlop: Occupation Mortality); vol. vii, p. 357 (J. C. Dunlop and R. M. Hunter: Some 1911 Census Results).

Census of Scotland, 1911, vol. iii, pp. xxiv-xxxix (Cd. 7163, On Fertility).

L. Le Mesurier: "Common-sense Economics" (John Murray).

Hartley Withers: "The Meaning of Money" (Smith Elder & Co.).

Assurance Companies Act, 1909, and Order of Board of Trade thereunder (*J.I.A.*, vol. xliv, p. 462).

Friendly Societies Acts, 1896 and 1908, and Collecting Societies and Industrial Assurance Companies Act, 1896 (actuarial and financial aspects only).

Income Tax Act, 1918: Sections 32, 33 and 46: Schedule D, Cases I and II, Rule 15; Case III, Rule 3; Case IV, Rules 1 and 2; Case V, Rules 1, 2 and 3; General Rules applicable to Schedules A, B, C, D and E: Nos. 19 and 21; Finance Act, 1920-21, Section 32.

The Valuation Forms and Instructions issued by the Registrar of Friendly Societies.

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March 1922.

ERRATA, *J.I.A.*, Vol. liii.

Page 183, 5th line from bottom,	for	slip 3	read	p. 172.
,, 195, line 11,	,,	$x^{(2k+2)}$	,,	$x^{[2k+2]}$.
,, 197, ,, 1,	,,	$\phi(x) > 70$	,,	$\phi(x) > 0$.

INDEX TO LEGAL NOTES

included in Vol. LIII *of the Journal.*

TABLE OF CASES REVIEWED

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OF THE

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